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Article · June 2021

DOI: 10.12345/22266070_2021_2_54

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UDC:517.957
DOI: 10.12345/22266070_2021_2_54

PLANE WAVES ON SHALLOW POLLUTED WATERS

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In this work, using inverse scattering techniques, the evolution of the single-soliton solution of the Korteweg-de Vries equation in the presence of perturbations is analyzed. The evolution of the scattering data is found for the Sturm-Liouville operator, the potential of which is a solution to the perturbed Korteweg-de Vries equation.The results obtained are illustrated with an example.The results can be applied to studying the process of wave propagation on shallow polluted waters.

Key words: shallow polluted waters, perturbed Korteweg-de Vries equation, inverse scattering method, single-soliton solution.

ПЛОСКИЕ ВОЛНЫ НА МЕЛКИХ ЗАГРЯЗНЕННЫХ ВОДАХ

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В данной работе с помощью метода обратной задачи рассеяния проанализирована эволюция односолитонного решения уравнения Кортевега-де Фриза при наличии возмущений. Найдена эволюция данных рассеяния для оператора Штурма-Лиувилля, потенциал которого является решением возмущенного уравнения Кортевега-де Фриза. Полученные результаты проиллюстрированы примерами. Полученные результаты могут быть применены для исследования процесса распространения волны на мелких загрязненных водах.

Ключевые слова: мелкие загрязненные воды, возмущенное уравнение Кортевега-де Фриза, метод обратной задачи, односолитонное решение.

ҰСАҚ ЛАСТАНҒАН СУЛАРДАҒЫ ЖАЗЫҚ ТОЛҚЫНДАР

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Бұл жұмыста кері шашырау есебінің әдісін қолдана отырып, бұзылулар болған кезде Кортевег-де-Фриз теңдеуінің бірсолитті шешімінің эволюциясы талданды. Штурм-Лиувилл операторы үшін шашырау деректерінің эволюциясы табылды, оның потенциалы кортевег-де-Фриздің бұзылған теңдеуінің шешімі болып табылады. Алынған нәтижелер мысалдармен *суреттелген. Алынған нәтижелер толқындардың ұсақ ластанған суларда таралу процесін зерттеу үшін қолданылуы мүмкін.*

Түйінді сөздер: ұсақ ластанған сулар, Кортевег-де-Фриздің бұзылған теңдеуі, кері есеп әдісі, бір политонды шешім.

1. Introduction

The Korteweg-de Vries (KdV) equation is the well-known model that was originally derived to describe shallow water waves of long wavelength and small amplitude. It is widely used in the theory of long waves, where it describes astonishingly well the main properties of nonlinear waves, even when their amplitudes are not small. In the derivation, the KdV equation assumed that all motion is uniform in all directions, along the crest of the wave. In that case, the surfaceelevation (above the equilibrium level h) of the wave, propagating in the x-direction, is a functiononly of the horizontal position x and of time t, i.e., $\eta = \eta(x,t)$.

In terms of the physical parameters, the KdV equation has the form

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{h}}\frac{\partial}{\partial x}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial x^2}\right),$$

where $\sigma = \frac{1}{3}h^3 - \frac{Th}{\rho g}$, h is the uniform water depth, ρ is the density of the water, g is the gravitational acceleration, α - small arbitrary constant and T stands for the surface tension.

In the Aral Sea water, there is a pollution of the water caused by the sold and the mud, which effects to the density of the water. As a result, the density of the water is not constant.So, $\rho = \rho(x,t)$. We assume $\rho_x << 1$.

By rescaling

$$t \to \frac{1}{2}\sqrt{\frac{g}{h}}\int_0^t \frac{d\tau}{\sqrt{\sigma}},\ x \to -\int_0^x \frac{dx}{\sqrt{\sigma}},\ \eta \to -2u - \frac{2}{3}\alpha,$$

we get

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = \varepsilon\frac{\partial^2 u}{\partial x^2}. \qquad (1)$$

Here ε is small quantity and $\varepsilon > 0$.

The KdV equation is completely integrable and possesses many remarkable properties, which can be found in the references cited below. In 1967, Gargner, Green, Kruskal, and Miura [1, pp. 1095-1097] had proposed the method of the inverse scattering problem for the Sturm-Liouville equation as a method for solving the Cauchy problem for the Korteveg-de Vriesequation

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

Shortly thereafter, Lax (see [2, pp. 467-490]) noted the general character of the method. Thus, a way was found for construction of several other classes of equations that can be solved by similar methods.

In the works [4-6], the Korteweg-de Vries equation and the modified Korteweg-de Vries equation with the right-hand side in various classes of functions were studied in detail.

The single-soliton solution to this equation (1) takes the form

$$u(x,t) = -2\chi^2 \operatorname{sech}^2 \chi(x - 4\chi^2 t).$$

Similarly, $-6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}$ expression can be written as the commutator of the following operators $L = -D^2 + u$ and $B = -4D^3 + 6uD + 3\frac{\partial u}{\partial x}$, that is

$$\frac{\partial u}{\partial t} - [L,B] = \varepsilon\frac{\partial^2 u}{\partial x^2}.$$

Here $D = \frac{\partial}{\partial x}$.Operator L satisfies the Sturm-Liouville equation $Ly = k^2 y$.

Using inverse scattering techniques, the evolution of the single-soliton solution of (1) in the presence of perturbations can be analyzed. The method, developed by Karpman and Maslov [3, pp. 281-291], is based on perturbation of the scattering data of the Sturm-Liouville equation for the KdV equation.

2. Scattering problem

Consider the Sturm-Liouville equation

$$Ly = -y'' + u(x)y = k^2 y, -\infty < x < \infty, \qquad (2)$$

with potential function $u(x)$, satisfying the condition of "rapidly decreasing"

$$\int_{-\infty}^{\infty} (1+|x|)|u(x)|dx < \infty. \qquad (3)$$

The present section contains information on the direct and inverse scattering problem for the problem (2)-(3), which is necessary for our further statement. The condition (3) provides that equation (2) possesses the Jost solutions $f(x,k)$ and $g(x,k)$ with the following asymptotic formulas:

$$\lim_{x\to\infty} f(x,k)\exp(-ikx) = 1, \ \lim_{x\to-\infty} g(x,k)\exp(ikx) = 1, \quad \operatorname{Im} k = 0.$$

The Jost solutions $f(x,k)$ and $g(x,k)$ admit an analytic continuation into the upper half-plane $\operatorname{Im} k > 0$ via variable k.

When k is real, the pairs $\{f(x,k), \ f(x,-k)\}$ and $\{g(x,k), \ g(x,-k)\}$ are pairs of linearly independent solutions for equation (2). Hense,

$$f(x,k) = b(k)g(x,k) + a(k)g(x,-k).$$

Where

$$a(k) = -\frac{1}{2ik} W\{f(x,k), \ g(x,k)\}. \qquad (4)$$

The function $a(k)$ admits an analytic continuation into the upper half-plane $\operatorname{Im} k > 0$ and has a finite number of simple zeroes $k_n = i\chi_n, \ (\chi_n > 0)$, $n = 1, 2, \dots, N$, meanwhile $-\chi_n^2$ is an eigenvalue of L;

For real k the below equality holds

$$g(x,k) = -b(-k)f(x,k) + a(k)f(x,-k).$$

According to representation (4),

$$g(x,i\chi_j) = B_j f(x,i\chi_j), \ \ j = 1, 2, \dots, N.$$

It's easy to check that, the functions

$$h_n(x) = \frac{\left.\frac{d}{dk}\big(g(x,k) - B_n f(x,k)\big)\right|_{k=i\chi_n}}{\dot{a}(i\chi_n)}, \qquad (5)$$

are the solutions of the equations $Ly = -\chi_n^2 y$. By equality (5), we define the following asymptotic formulas:

$$h_n(x) \to \exp(\chi_n x), x \to \infty,$$

$$h_n(x) \to -B_n \exp(-\chi_n x), x \to -\infty.$$

The set $\{r^+(k), \ \chi_1, \chi_2, \dots, \chi_N, B_1, B_2, \dots, B_N\}$ is called scattering data for the problem (2)-(3).

Let the function $u(x,t)$ in (2) be a solution to the KdV equation with the right-handside

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = G(x,t), \qquad (6)$$

where the function $G(x,t)$ - is sufficiently smooth and

$$G(x,t) = o(1), \ \ x \to \pm\infty, \ t \ge 0.$$

The equation (6) is considered with the initial condition

$$u(x,0) = u_0(x),$$

where initial function $u_0(x)$ satisfies the following properties:

1. $\int\limits_{-\infty}^{\infty}(1+x^2)\left|u_0(x)\right|dx<\infty,$

2. The equation $-y''+u_0(x)y=k^2y,\ x\in R^1$ has exactly N number of negative eigenvalues $-\chi_1^2(0), -\chi_2^2(0), ..., -\chi_N^2(0)$.

Let us assume that the function $u(x,t)$ is sufficiently smooth and it tends to its limits rapidly enough when $x\to\pm\infty$ and satisfy the condition

$$\int\limits_{-\infty}^{\infty}\left(\left(1+|x|\right)\left|u(x,t)\right|+\sum_{j=1}^{3}\left|\frac{\partial^j u(x,t)}{\partial x^j}\right|\right)dx<\infty\ , \tag{7}$$

for all $t\geq 0$.

As it is shown in [7, pp. 488-492] and [8, pp.1341–1356], the following Theorem is valid.

Theorem. If the potential function $u(x,t)$ is a solution of the equation (6) in the class of functions (7), then the scattering data of the problem (2) with the function $u(x,t)$ depend on *t* as follows:

$$\frac{dr^+}{dt}=8ik^3\,r^+-\frac{1}{2ika^2(k)}\int\limits_{-\infty}^{\infty}G(x,t)g^2dx\,,\ \mathrm{Im}k=0\,,$$

$$\frac{dB_n}{dt}=8\chi_n^3B_n-\frac{1}{2\chi_n}\int\limits_{-\infty}^{\infty}G(x,t)g(x,i\chi_n,t)h_n(x,t)dx,$$

$$\frac{d\chi_n}{dt}=-\frac{1}{2\chi_n}\frac{\int\limits_{-\infty}^{\infty}Gg^2(x,i\chi_n,t)dx}{\int\limits_{-\infty}^{\infty}g^2(x,i\chi_n,t)dx}\,,\ n=1,2,...,N\,. \tag{8}$$

3.Results

Let us apply the result of theorem for the case $G(x,t)=\varepsilon\dfrac{\partial^2 u}{\partial x^2}$ and $u(x,t)=-2\chi^2\,\mathrm{sech}^2\,z$, where $z=\chi(x-\xi)$. Now, we determine the dependence of the parameters of the soliton χ and ξ on time. In this case, the Jost solution of

$$-y''-2\chi^2\,\mathrm{sech}^2\,zy=k^2y,\ \ x\in R$$

has the form

$$f(x,k,t)=\frac{k+i\chi thz}{k+i\chi}e^{ikx}, \tag{9}$$

$$g(x,k,t)=\frac{k-i\chi thz}{k+i\chi}e^{-ikx}. \tag{10}$$

When $k=i\chi$, we have

$$f(x,i\chi,t)=\frac{1}{2}e^{-\chi\xi}\,\mathrm{sech}\,z\,,\ g(x,i\chi,t)=\frac{1}{2}e^{\chi\xi}\,\mathrm{sech}\,z\,.$$

Accordingly,

$$B_1=\frac{g(x,i\chi,t)}{f(x,i\chi,t)}=e^{2\chi\zeta}\,. \tag{11}$$

The perturbation of the eigenvalue is obtained from (8), which takes the form

$$\frac{d\chi}{dt}=-\frac{1}{2\chi}\frac{\int\limits_{-\infty}^{\infty}Gg^2(x,i\chi,t)dx}{\int\limits_{-\infty}^{\infty}g^2(x,i\chi,t)dx}=-\frac{\varepsilon}{4\chi}\int\limits_{-\infty}^{\infty}u_{xx}\sec h^2zdz\,. \tag{12}$$

Using $\int_{-\infty}^{\infty} \operatorname{sech}^4 z dz = \frac{4}{3}$ and $\int_{-\infty}^{\infty} \operatorname{sech}^6 z dz = \frac{16}{15}$ the equality(12)can be written as

$$\frac{d\chi}{dt} = -\frac{8}{15}\varepsilon\chi^3. \qquad (13)$$

The dependence of the phase term on time $\xi(t)$ is obtained from the dependence on time $B_1(t)$ and relation (11):

$$\frac{dB_1}{dt} = \left(2\xi\frac{d\chi}{dt} + 2\chi\frac{d\xi}{dt}\right)e^{2\chi\xi}.$$

Using (9), (10) and (5), we get

$$h_n(x) = ie^{\chi\xi}\left(\frac{1}{2}sh2z - \chi x\right)\sec hz,$$

$$\frac{dB_1}{dt} = 8\chi^3 B_1 - \frac{1}{2}e^{2\chi\xi}\int_{-\infty}^{\infty} G(x,t)\left(sh2z - 2z + 2\chi\xi\right)\sec h^2 z dz = e^{2\chi\xi}(8\chi^3 - \frac{32}{15}\xi\varepsilon\chi^5), \qquad (14)$$

From (14), we can write

$$\frac{d\xi}{dt} = 4\chi^2 - \frac{8}{15}\xi\varepsilon(2\chi^4 - \chi^2).$$

By integrating, we get the following expression

$$\xi = \xi_0\chi^4\chi_0^{-4}\exp(4\chi^2 - 4\chi_0^2) + 4\chi_0^4\int_0^t \chi^{-2}\exp(4\chi_0^2 - 4\chi^2)d\tau. \qquad (15)$$

By using the expressions for the χ and ξ, we can finally write the solution as

$$u(x,t) = -2\left(\chi_0^{-2} + \frac{4}{15}\varepsilon t\right)^{-1}\operatorname{sech}^2\sqrt{\left(\chi_0^{-2} + \frac{4}{15}\varepsilon t\right)^{-1}}(x -$$

$$-\xi_0\chi^4\chi_0^{-4}\exp(4\chi^2 - 4\chi_0^2) - 4\chi_0^4\exp(4\chi_0^2)\int_0^t \chi^{-2}\exp(-4\chi^2)dt). \qquad (16)$$

For example, we will solve the following Cauchy's problem:

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = \varepsilon\frac{\partial^2 u}{\partial x^2},$$

$$u(x,0) = -2\operatorname{sech}^2 x.$$

First, we find a solution of the direct problem for the following equation:

$$-y'' - 2\operatorname{sech}^2 x\, y = k^2 y, \quad x \in R.$$

As a result, we obtain the following Scattering Data

$$a(k,t) = \frac{k-i}{k+i}, \; N = 1, \; r^+(k,t) = 0, \; B_1 = 1, \; \chi_1 = 1.$$

Using the equations (13) and (15) we find of χ and ξ depending on t:

$$\chi(t) = \left(\frac{4}{15}\varepsilon t + 1\right)^{-\frac{1}{2}},$$

$$\xi(t) = \chi^4\exp(4\chi^2 - 4) + 4\int_0^t \chi^{-2}e^{4-4\chi^2}dt.$$

By using the expressions for the χ and ξ we can finally write the solution as

$$u(x,t) = -2\left(1 + \frac{4}{15}\varepsilon t\right)^{-1}\operatorname{sech}^2\sqrt{\left(1 + \frac{4}{15}\varepsilon t\right)^{-1}}(x - \chi^4 e^{4\chi^2 - 4} - 4\int_0^t \chi^{-2}e^{4-4\chi^2}d\tau).$$

4. Conclusion

In this paper, we have studied the properties of the soliton solution of the KdV equation which is used to describe plane waves on shallow polluted waters, as well as appearing in other applied areas. With the changes of variables, this equation has been reduced to the form (1). Using inverse scattering techniques, we have found the form of a solitary-wave solution to a KdV equation in the presence of perturbations. Whereas the unperturbed KdV equation has a soliton solution (16). Pollution affects the distribution of waves on shallow water. In the presence of an increase in density, the amplitude, and speed of wave distribution decrease.

REFERENCES:

1. **Gardner C.S., GrinI.M., Kruskal M.D., Miura R.M. Method for solving the Korteweg-de Vries equation** [Tekst]:/ GardnerC.S., GrinI.M., KruskalM.D., MiuraR.M. Phys. Rev. Lett., 1967. Vol. 19, pp. 1095-1097.
2. **Lax P.D. Integrals of nonlinear equations of evolution and solitary waves** [Tekst]:/ LaxP.D. Comm. Pure Appl. Math., 1968, No.21, pp. 467-490.
3. **Karpman V. I.,Maslov E. M. Perturbation theory for solitons** [Tekst]:/Karpman V. I.,MaslovE. M. Sov. Phys. JETP, 1977, 46(2), pp. 281-291.
4. **Babadjanova A., Kriecherbauer T., Urazboev G. The periodic solutions of the discrete modified KdV equation with a self-consistent source** [Tekst]:/ Babadjanova A., Kriecherbauer T., Urazboev G. Applied Mathematics and Computation, 2020, 376, pp. 125-136
5. **Urazboev G.U., Xoitmetov U.A., Babadjanova A.K. Integration of the Matrix Modified Korteweg-de Vries Equation with an Integral-Type Source** [Tekst]:/Urazboev G.U., Xoitmetov U.A., Babadjanova A.K. Theoretical and Mathematical Physics, 2020, 203 (3), pp.734-746
6. **Urazboev G.U., Babadjanova A.K. On the Integration of the matrix modified Korteweg-de Vries equation with a self-consistent source** [Tekst]:/ Urazboev G.U., Babadjanova A.K. Tamkang Journal of Mathematics, 2019, 3(50), pp.281-291
7. **Mel'nikov V.K. Exact solutions of the Korteweg-de Vries equation with a self-consistent source** [Tekst]:/Mel'nikov V.K.Physics Letters A, 1988, Vol. 128, Issue 9, pp. 488-492.
8. **Urazboev G.U., Khasanov A.B. Integrating the Korteweg–de Vries Equation with a Self-Consistent Source and “Steplike” Initial Data** [Tekst]:/ Urazboev G.U., Khasanov A.B. Theoretical and Mathematical Physics, 2001, Vol. 129, pp.1341–1356.

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