The dual surfaces of an isotropic space R_3^2

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R_3^2 Izotrop fazoda qo'shma sirt

Ushbu maqolada, izotrop fazoda qavariq sirtlarning ba'zi xossalari o'rganilgan. R_3^2 fazoning ta'rifi, birinchi va ikkinchi kvadratik formalar, hamda yevklid fazo sferasining analogi aniqlangan. Izotrop fazo sferasi aylanma paraboloid bo'ladi. Izotrop sferaning tekislik bilan kesimi ellips bo'lishi isbotlangan. Izotrop fazoda sferaga nisbatan qo'shma akslantirish ta'rifi berilgan. Qo'shma akslantirish yordamida sirtning qo'shma tasviri aniqlangan. Qo'shma sirtning bazi xossalari isbot qilingan.

<u>Kalit so'zlar:</u> Paraboloid; tekislik; izotrop fazo; qo'shma akslantirish; qo'shma sirt.

Двойственные поверхности изотропного пространства R_3^2

В работе изучается некоторые свойства выпуклых поверхностей в изотропном пространстве R_3^2. Дано определение R_3^2, первое и второе квадратичние формы поверхности и найдена аналог сферы евклидова пространства. Сферой изотропного пространства является параболоид врашения. Доказана что пересечение сферы изотропного пространства всегда является эллипсом. Дано определение двойственного образа плоскости относительно сферы изотропного пространства. С помошю двойственного образа определена обобшенного отображения поверхности. Доказана некоторие свойства двойственной поверхности.

<u>Ключевые слова:</u> Параболоид; плоскость; изотропное пространство; двойственные отображение; двойственные поверхность.

Isotropic space

Consider a three-dimensional affine space A_3. Let $X\{x_1,y_1,z_1\}$ and $Y\{x_2,y_2,z_2\}$ be vectors of the space A_3 in the coordinate system $O\{e_1,e_2,e_3\}$.

We say the scalar product of vectors X and Y the number defined by the following rule:

$$(X,Y)_1 = x_1x_2 + y_1y_2 \tag{1}$$

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in the case of $(X, Y)_1 \neq 0$ and

$$(X, Y) = z_1 z_2$$

if $(X, Y)_1 = 0$.

Definition 1. *The affine space A_3, in which the scalar product of vectors is calculated by formula (1), is called the isotropic space R_3^2.*

An isotropic space is a representative of semi-euclidean spaces [5, 2].

The norm of the vector $|\vec{x}|$ is defined as the square root of the scalar square of this vector, and the distance between two points is defined as the norm of the vector connecting these points [5].

Thus, the distance between points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in R_3^2 is calculated by the formula

$$|\vec{AB}| = (AB)_1 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{2}$$

if $(AB)_1 \neq 0$, and $|\vec{AB}| = (AB)_2 = |z_2 - z_1|$, if $(AB)_1 = 0$.

If one uses the superimposed space method, i.e. the points and vectors of the space are considered, respectively, as points and vectors of the Euclidean space R_3, then the distance determined by formula (2) will be the length of the projection of the Euclidean segment AB onto the plane Oxy, parallel to the axis Oz. If $(AB)_1 = 0$, then points lie on a straight line parallel to the projection, and the second distance $(AB)_2$ coincides with the Euclidean distance between the points A and B.

Obviously, when $(AB)_2 = 0$ too, the points A and B coincide.

The motion, i.e. a bijection of the space R_3^2 onto itself, preserving the distance $|\vec{AB}|$, is given by the formula [2, 12, 8]:

$$\begin{cases} x' = x \cos\varphi - y \sin\varphi + a, \\ y' = x \sin\varphi + y \cos\varphi + b, \\ z' = \alpha x + \beta y + z + c. \end{cases} \tag{3}$$

It is easy to see that the motion of R_3^2 consists of a parallel translation, a rotation around the coordinate line Oz, and a sliding (a parabolic rotation), in which straight lines, parallel to the axis Oz, slide along themselves.

The sphere of the isotropic space R_3^2, , that is, the set of points in the space equidistant from one point, has the equation

$$(x - x_0)^2 + (y - y_0)^2 = r^2, \tag{4}$$

where (x_0, y_0, z_0) is its center and r is its radius. Naturally, the center of the sphere is a straight line passing through the point $(x_0, y_0, 0)$ and parallel to the axis Oz.

A spere in R_3^2 is affinely equivalent to a cylinder whose directrix is a circle on the plane Oxy with generatrices parallel to the axis Oz. The symmetry axis of the cylinder forms the center of the sphere of the isotropic space R_3^2.

According to equality (2), the geometry on the plane $z = 0$ is Euclidean. It is easy to prove that the geometry on the planes of an isotropic space, that uniquely projected onto the plane $z = 0$, will also be Euclidean.

In all planes passing through the axis Oz, or parallel to it, the geometry will be Galilean [2].

A motion (3) of the isotropic space R_3^2 does not transfer Euclidean planes to Galilean planes and vice versa.

The isotropic space R_3^2 is an affine space, therefore, when moving, the parallelism of vectors and planes is preserved. If the vectors X and Y are parallel to an Euclidean plane and are not collinear, then the abscissas and ordinates of these vectors are not proportional. When vectors are parallel to the Galilean plane, their abscissas and ordinates are proportional, and they differ in projections onto the axis Oz.

The angle between the vectors X and Y is defined as the angle between the vectors of the Euclidean or Galilean plane, depending on which plane these vectors are parallel. Therefore, the angle between the vectors $\vec{X}\{x_1, y_1, z_1\}$ and $\vec{Y}\{x_2, y_2, z_2\}$ is defined by the formula

$$\cos\varphi = \frac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}; \tag{5}$$

in the case of $\varphi = 0$, the angle between the vectors is defined by the formula

$$h = \frac{1}{\sqrt{x_1^2 + y_1^2}}|z_2 - z_1|. \tag{6}$$

The geometric meaning of the angles φ and h can be interpreted with the help of the unit vectors

$$\tilde{X}\left\{\frac{x_1}{\sqrt{x_1^2+y_1^2}}, \frac{y_1}{\sqrt{x_1^2+y_1^2}}, \frac{z_1}{\sqrt{x_1^2+y_1^2}}\right\}$$

and

$$\tilde{Y}\left\{\frac{x_2}{\sqrt{x_1^2+y_1^2}}, \frac{y_2}{\sqrt{x_1^2+y_1^2}}, \frac{z_2}{\sqrt{x_1^2+y_1^2}}\right\}$$

directed respectively along $\vec{X}$ and $\vec{Y}$.

The angle φ is the usual Euclidean angle between the projections of the vectors $\vec{X}$, $\vec{Y}$ on the plane Oxy. If the projections of the vectors on the plane Oxy coincide, then $\varphi = 0$, and h is the Galilean angle, wich is equal to the Euclidean distance between the ends of the vectors $\vec{X}$ and $\vec{Y}$ on the generatrix of the cylinder $x^2 + y^2 = 1$.

Basic concepts of the surfaces theory of an isotropic space

Let a surface F be given by the equation

$$z = f(x,y), \quad (x,y) \in D, \tag{7}$$

where $f(x,y) \in C^3(D)$, and D is a domain on the plane Oxy of the isotropic space R_3^2.

Consider a curve γ on the surface F. If, by analogy with the Euclidean space, we define the first quadratic form of a surface, then it has the form[3]

$$I = dx^2 + dy^2. \tag{8}$$

Obviously, this is a degenerate quadratic form of the surface F.

The considered surface F is uniquely projected onto the domain D of the plane Oxy. By calculating the angle between the curve and the area of the domain of the surface, which are related by the first quadratic form (8), one can make sure that they are equal to the corresponding projections on the plane Oxy.

By analogy with the Euclidean space, we define the second quadratic form of a surface in an isotropic space.

The concept of a tangent plane is affine, hence the tangent plane of the surface F of the isotropic space R_3^2 does not differ from the tangent plane of the same surface, considered in Euclidean space.

In an isotropic space, for any vector lying on the tangent plane of the surface M, a vector, directed along the axis Oz, is orthogonal in the sense of the isotropic space. Therefore, a vector, collinear to the direction of Oz, is called the normal of the tangent plane of the isotropic space R_3^2.

The plane of normal section, i.e. a plane passing along the normal to the surface and the tangent curve, is always the Galilean plane.

The second quadratic form of surface (7) has the form

$$II = z_{xx}dx^2 + 2z_{xy}dxdy + z_{yy}dy^2. \tag{9}$$

The normal curvature of the curve γ on the surface F, that is the curvature of the curve of the normal section, is defined as the curvature of the curve on the Galilean plane ([2])

$$k_n = \frac{II}{I}.$$

There are two types of spheres in an isotropic space. The first of them, as the locus of points equidistant from a given point, is defined by formula (4). The second type of sphere is defined as a surface with constant normal curvature in all directions on the surface. This kind of sphere is analogous to the Euclidean sphere.

Lemma 1. *The normal curvature of a curve on the surface*

$$2 \cdot z = x^2 + y^2 \tag{10}$$

is constant in all directions and is equal to unity.

The proof of Lemma 1 is reduced to directly calculating the normal curvature of a curve on surface (10).

By analogy with the Euclidean space, points on a surface are called umbilical if the normal curvature in all directions is equal.

According to Lemma 1, all points of surface (10) are umbilical.

Therefore, surface (10) is called the second sphere or isotropic sphere. The value of the inverse to the normal curvature is called the radius of the sphere. Therefore sphere (10) will be a unit isotropic sphere.

Geometric meaning of dual mapping

In ([1]), a cylindrical mapping of a surface in semi-Euclidean spaces is constructed, a particular case of which is the considered isotropic space R_3^2.

An isotropic space is a self-dual projective space ([2]). This means that the projective dual space to the isotropic space R_3^2 is also an isotropic space.

In the case of self-dual spaces, the cylindrical map of the surface coincides with the dual surface with respect to the sphere of the space under consideration [4].

Let us find out the geometric meaning of the dual correspondence in an isotropic space R_3^2.

Consider an isotropic sphere of unit radius given by Equation (10) and the plane

$$z = Ax + By + C \tag{11}$$

intersecting this isotropic sphere.

Theorem 1. *The set of intersection points of sphere (10) with plane (11) coincides with the points of tangency of the cone given by the equation*

$$(x - A)^2 + (y - B)^2 - (z + C)^2 = 0 \tag{12}$$

with sphere 10.

Proof. Equations of the section of sphere (10) with plane (11) has the form

$$\frac{1}{2}(x^2 + y^2) = Ax + By + C$$

If we reduce it to the canonical form, then it will take the form

$$(x - A)^2 + (y - B)^2 = A^2 + B^2 + 2C \tag{13}$$

The locus of the points of the isotropic space R_3^2, satisfying Equation (13), belongs to the cylinder centered at the point $(A, B, 0)$ and the radius $r = \sqrt{A^2 + B^2 + 2C}$, as well as unit sphere (10). Moreover, the condition $A^2 + B^2 + 2C > 0$ coincides with the condiition of intersection of plane (11) with the spere of the isotropic space. The generatrices of the cylinder are parallel to the axis Oz. The straigt line, passing through the point $(A, B, 0)$ and parallel to the axis Oz, is the axis of the cylinder.

Consider an arbitrary plane π passing along the axis of the cylinder l. The geometry on the plane π will be Galilean. The section of the plane π by isotropic sphere (10) will be a parabola, i.e. a circle of the corresponding Galilean plane.

Denote the line formed by the intersection of the sphere and plane (11) by L. Draw from each point of L tangent planes to the sphere. Denote the set of these planes by $\{\alpha\}$. □

Theorem 2. *All planes from $\{\alpha\}$ intersect at one point.*

Proof.

Consider three points belonging to the sphere and plane (11), that is, points on the curve $L : M(x_1, y_1, z_1)$, $N(x_2, y_2, z_2)$, and $P(x_3, y_3, z_3)$. Draw through these points tangent planes to the sphere. The equation of the tangent plane passing through the points M, N, P has the following form:

$$-x_1(x - x_1) - y_1(y - y_1) + z - z_1 = 0,$$

$$-x_2(x - x_2) - y_2(y - y_2) + z - z_2 = 0,$$

$$-x_3(x - x_3) - y_3(y - y_3) + z - z_3 = 0.$$

The coordinates of the points of intersection of these planes are calculated by the formula:

$$\begin{cases} \xi = \dfrac{(y_2 - y_1)(z_1 - z_3) - (z_1 - z_2)(y_3 - y_1)}{(y_3 - y_1)(x_2 - x_1) - (y_2 - y_1)(x_3 - x_1)}, \\ \eta = \dfrac{(x_2 - x_1)(z_1 - z_3) - (z_1 - z_2)(x_3 - x_1)}{(x_3 - x_1)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_1)}, \\ \zeta = x_1\xi + y_1\eta - z_1. \end{cases}$$

It is known that the points M, N, P lie in the same plane. The equation for this plane can be written as follows:

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

where

$$\begin{cases} a = (y_2 - y_1)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_1), \\ b = (z_2 - z_1)(x_3 - x_1) - (x_2 - x_1)(z_3 - z_1), \\ c = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1). \end{cases}$$

We obtain from here

$$\begin{cases} \xi = -\dfrac{a}{c}, \\ \eta = -\dfrac{b}{c}, \\ \zeta = -\dfrac{a}{c}x_1 - \dfrac{b}{c}y_1 - z_1. \end{cases}$$

Take an arbitrary point $Q(x_4, y_4, z_4)$, on the curve L.

Then the equation of the tangent plane passing through the point Q,

$$-x_4(x - x_4) - y_4(y - y_4) + z - z_4 = 0.$$

We must show that the point $\{\xi, \eta, \zeta\}$ belongs to this plane. We have

$$-x_4\left(-\frac{a}{c} - x_4\right) - y_4\left(-\frac{b}{c} - y_4\right) + \left(-\frac{a}{c}x_1 - \frac{b}{c}y_1 - z_1\right) =$$

$$= \frac{a}{c}x_4 + \frac{b}{c}y_4 - z_4 + x_4^2 + y_4^2 - z_1 - \frac{a}{c}x_1 + \frac{b}{c}y_1 =$$

$$= (x_4 - x_1)\frac{a}{c} + (y_4 - y_1)\frac{b}{c} - z_4 + 2z_4 - z_1 =$$

$$= \frac{1}{c}\left[a(x_4 - x_1) + b(y_4 - y_1) + c(z_4 - z_1)\right] = 0.$$

Hence, the coordinates of the intersection point of the three planes also satisfy the equation of the tangent plane passing through Q. Since we have chosen an arbitrary point of the curve L as the point Q, we can conclude that all tangent planes touching the sphere at the points of the curve L, have one common point.

Comparing the equations of the plane passing through the points M, N, P and the points of intersection of all possible tangent planes from $\{\alpha\}$, we obtain

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0,$$

$$z = \frac{a}{c}x - \frac{b}{c}y + \frac{a}{c}x_1 + \frac{b}{c}y_1 - z_1.$$

Taking into account that this equation coincides with equation (11), we have:

$$\begin{cases} A = -\dfrac{a}{c}, \\ B = -\dfrac{b}{c}, \\ C = \dfrac{a}{c}x_1 + \dfrac{b}{c}y_1 - z_1, \end{cases} \qquad \text{hence} \qquad \begin{cases} \xi = A, \\ \eta = B, \\ \zeta = -C. \end{cases}$$

Theorem 2 is proved.   □

As a corollary of this theorem, we can assert that the cone with the apex at the point $(A, B, -C)$ and the directrix L is the tangent cone of the sphere (10).

It is known [4], under projective duality, a point corresponds to a plane of the dual space, and vice versa, a plane corresponds to a point. From the above proved theorem, it follows that the plane given by equation (11) can be associated with the point $(A, B, -C)$, and we can call it dual to this plane with respect to the sphere (10). On the other hand, there is such correspondence with respect to the sphere of the Euclidean space [11].

Definition 2. *The point $(A, B, -C)$ is said to be the dual image of the plane $z = Ax + By + C$ with respect to the sphere $z = \frac{1}{2}(x^2 + y^2)$.*

Dual mapping of a surface

Let the curve L be the intersection of the sphere (10) with the plane $z = H$, and its projection Γ on the plane $z = 0$ bound the domain D. It is easy to prove that L and its projection Γ are circles centered at the oriigin.

Consider the set of surfaces given by the equation $z = f(x, y)$, $(x, y) \in D$ with the border L and contained within the sphere (10). The interior of the sphere is considered to be a part of the space R_3^2 satisfying the condition $z > \frac{1}{2}(x^2 + y^2)$ (Fig.1).
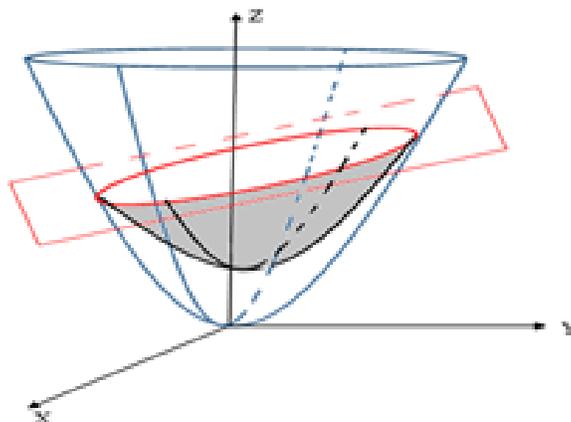
Further, let the discriminant of the second quadratic form of the surface (7) be strictly positive: $z''_{xx} \cdot z''_{yy} - z''_{xy}{}^2 > 0$ and $z''_{xx} > 0$. Then the surface F is a convex surface with a boundary projecting onto the border Γ of the domain D and is convex downward [9].

Consider the tangent (in the case of irregularity, support) plane π_0 at the point $M \in F$, and let $M'(x_0, y_0) \in D$ be the projection of M.

The equation of the tangent plane to F at the point M has the following form:

$$z - z_0(x_0, y_0) = f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0).$$

Let us give it to the form

$$z = f'_x(x_0, y_0) \cdot x + f'_y(x_0, y_0) \cdot y + z_0(x_0, y_0) - f'_x(x_0, y_0) \cdot x_0 - f'_y(x_0, y_0) \cdot y_0. \tag{14}$$

Since the considered surface is uniquely projected onto the plane Oxy, the equation of the tangent plane can be written in the form (14).

Tangent plane (14) dual corresponds to the point M^* in the space R_3^2 with coordinates

$$M^* \left(f'_x(x_0, y_0), \ f'_y(x_0, y_0), \ f'_x(x_0, y_0) \cdot x_0 + f'_y(x_0, y_0) \cdot y_0 - z_0(x_0, y_0) \right).$$

Lemma 2. *The surface F, satisfying the above conditions, has a unique minimum in the domain D.*

Proof.

The proof of Lemma 2 follows from the unique projection and strict downward convexity of the surface F [9].

Let $U \subset F$ be a neighborhood of the point F, and $U' \subset D$ be the set of points M' which are projections of the point M on the plane D.

In the general case, there is a certain support plane $\pi(M)$ at each point $M \in F$, and a certain point M^* corresponds dually to this support plane $\pi(M)$.

All possible support planes of points of the surface F correspond to a set of points M^*, which in the general case form a surface F^* in R_3^2.   □

Definition 3. *The surface F^* is said to be the dual surface of the surface F in the isotropic space R_3^2.*

Theorem 3. *The dual image of the minimum of the surface F is the intersection point of F^* with the coordinate axis Oz, and, conversely, the intersection point of F with Oz corresponds to the minimum point of F^*.*

Proof. Let $(0, 0, z_0)$ be the intersection point of F with the axis Oz. Then the dual image of the tangent plane will be the point M^* with the coordinates $(f'_x(0,0), f'_y(0,0), -f(0,0))$.

Suppose that M^* is not the minimum of the dual surface F^*. Then there must be a point $(A_0, B_0, -C_0)$, which is the minimum of F^*. This means that there exists the tangent plane $z = A_0 x + B_0 y + C_0$ for F, moreover $C_0 \geq f(0,0)$.

But this can only be done in the case of $C_0 = f(0,0)$, hence, $f'_x(0,0) = 0$, $f'_y(0,0) = 0$, and $C_0 = f(0,0)$, and this it possible only when when the minima of the surfaces are on the axis Oz.

If $f'_x(0,0) \neq 0$, $f'_y(0,0) \neq 0$, then there cannot be $C_0 > f(0,0)$ since the intersection point of the tangent corresponding to the minimum of the surface F^* cannot Oz inside the surface F. This contradiction shows that the point $M^*(f'_x(0,0), f'_y(0,0), -f(0,0))$ is the minimum of F^*.

Let us prove the second part of Theorem.

Let $M(a, b, f(\,)a, b))$ be the minimum point of F. Then the minimum condition implies that

$$f'_x(0,0) = 0, \quad f'_y(0,0) = 0.$$

Hence, the dual image of the tangent point of the minimum point has the coordinates

$$M^*(0, 0, -f(a,b)).$$

The fact that this point belongs to Oz, is obvious.

Theorem is proved.   □

Theorem 4. *The surface F^* is uniquely projected onto the plane Oxy, in this case F^* is convex downward if the surface F is convex downward.*

Proof.

Take two arbitrary points P, Q of F, and let $\alpha(P)$, $\beta(Q)$ be the support planes at these points of the surface.

Denote by P^* and Q^* the dual images of the support planes $\alpha(P)$, $\beta(Q)$. Of course, they belong to the dual surface F^*. Draw the straight line l^* through the points P^* and Q^*. Suppose that F^* is nonconvex. Then one can find two points belonging to F^* such that the straight line, passing through these points, intersects F^* at least st three points. If these points are P^* and Q^*, then there exists a point $D^* \in F^*$ and $D^* \in l^*$. Then there is a point D on the surface F being the preimage of the point D^*.

But according to the duality condition, D must belong to the straight line l passing through the points P, Q. This is the convexity condition for F. The resulting contradiction shows that any line passing through two points of F^*, intersects it only at these points. Therefore, F^* is convex.

The proof of downward convexity follows from the above mentioned Theorem.   □

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