Chapter 13
Time-Independent Nonlinear Schrödinger Equation on Simplest Networks

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Abstract We treat the time-independent (cubic) nonlinear Schrödinger equation (NLSE) on simplest networks. In particular, the solutions are obtained for star and tree graphs with the boundary conditions providing vertex matching and flux conservation. It is shown that the method can be extended to the case of arbitrary number of bonds in star graphs and for other simplest topologies.

13.1 Introduction

The nonlinear evolution equations were the topic of extensive research during the last half century. Special attention among others has attracted nonlinear Schrodinger equation whose detailed treatment started in the pioneering studies by Zakharov and Shabat in early seventies of the last century [1–3]. Such an interest is mainly caused by the possibility for obtaining soliton solution of NLSE and its various practical applications in different branches of physics. Initially, the applications of NLSE and other nonlinear evolution equations having soliton solutions were mainly focussed in optics, acoustics, particle physics, hydrodynamics and biophysics. However, special attention NLSE and its soliton solutions have attracted because of the recent

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progress made in the physics and Bose-Einstein condensates (BEC). Namely, due to the fact that the dynamics of BEC is governed by Gross-Pitaevski equation which is NLSE with cubic nonlinearity, finding the soliton solution of NLSE with different confining potentials and boundary conditions is of importance for this area of physics.

During the last few decades NLSE and its soliton solutions have been treated in the context of fiber optics, photonic crystals, acoustics BEC and other topics (see books [4–8] and references therein). Both, stationary and time-dependent NLSE were extensively studied for different trapping potentials in the context of BEC. In particular, the stationary NLSE was studied for box boundary conditions [9, 10] and the square well potential [11–14].

In this work we explore the time-independent NLSE on networks by modeling the latter by graphs. Graphs are the systems consisting of <u>bonds</u> which are connected at the <u>vertices</u> [15]. The bonds are connected according to a rule that is called topology of a graph. Topology of a graph is given in terms of so-called adjacency matrix which can be written as [16, 17]:

$$C_{ij} = C_{ji} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are connected,} \\ 0 & \text{otherwise,} \end{cases} \qquad i, j = 1, 2, \ldots, V.$$

Earlier, the linear Schrödinger equation on graphs was treated in different contexts (e.g., see reviews [16–18] and references therein). In this case the eigenvalue problem is given in terms of the boundary conditions providing continuity and current conservation [16–22].

Despite the progress made in the study of the linear Schrödinger equation on graphs, corresponding nonlinear problem, i.e., NLSE on graphs is still remaining as less-studied topic. This is mainly caused by difficulties that appear in the case of NLSE on graphs, especially, for the time-dependent problem. In particular, the problem becomes rather nontrivial and it is not so easy to derive conservation laws [28]. However, during the last couple of years there were some attempts to treat time-dependent [28, 29] and the stationary [30, 32] NLSE on graphs. Soliton solutions and connection formulae are derived for simple graphs in the Ref. [28]. The problem of fast solitons on star graphs is treated in the Ref. [29]. In particular, the estimates for the transmission and reflection coefficients are obtained in the limit of very high velocities. The problem of soliton transmission and reflection is studied in [30] by solving numerically the stationary NLSE on graphs.

In [31] dispersion relations for linear and nonlinear Schrödinger equations on networks are discussed. More recent treatment of the stationary NLSE in the context of scattering from nonlinear networks can be found in the Ref. [32]. In particular, the authors discuss transmission through a complex network of nonlinear one-dimensional leads and found the existence of the high number of sharp resonances dominating in the scattering process. The stationary NLSE with power focusing nonlinearity on star graphs was studied in very recent paper [33], where existence of the nonlinear stationary states are shown for $\delta-$type boundary conditions. In particular, the authors of [33] considered a star graph with N semi-infinite bonds, for which they obtain the exact solutions for the boundary conditions with $\alpha \neq 0$. The properties of the ground state wave function are also studied by considering separately the cases of odd and even N. In this work we treat NLSE on simplest graphs with finite-length bonds, aiming at obtaining its exact solutions for some types of the boundary conditions.

An important applications of NLSE on networks is Bose-Einstein condensation (BEC) and transport of BEC in networks. This issue has been extensively discussed recently in the literature [24–27]. We note that networks can be used as the traps for BEC experiments.

It is important to notice that earlier the problem of soliton transport in discrete structures and networks was mainly studied within the discrete NLSE [23]. However, such an approach doesn't provide comprehensive treatment of the problem and one needs to use continuous NLSE on graphs. The aim of this work is the formulation and solution of stationary NLSE on simplest graphs such as star, tree and loop which can be considered as exactly solvable topologies.

13.2 Time-Independent NLSE on Primary Star Graph

The problem we want to treat is the stationary (time-independent) NLSE with cubic nonlinearity on the primary star graph. The star graph is a three or more bonds connected to one vertex (branching point). The primary star graph consisting of three bonds, b_1, b_2, b_3, is plotted in Fig. 13.1. The coordinate, x_1 on the bond b_1 varies from 0 to L_1, while for the bonds $b_k, k = 2,3$ the coordinates, x_k, vary from L_1 to L_k. At the branching point we have $x_k = L_1$. In the following we will use the notation x instead of x_k, $(k = 1,2,3)$. Then the time-independent NLSE can be written for each bond as

$$-\psi_j'' \pm \beta_j|\psi_j|^2\psi_j = \lambda^2\psi_j,\ \beta_j > 0,\ j = 1,2,3. \tag{13.1}$$

Eq. (13.1) is a multi-component equation in which components are mixed through the boundary conditions and conservation laws. More detail analysis of the boundary conditions on graphs can be found in the Refs. [19, 22].

In this paper we will consider the following boundary conditions:

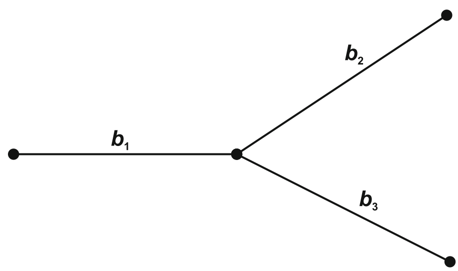


Fig. 13.1 Primary star graph consisting of three bonds

$$\psi_1(L_1) = A_2\psi_2(L_1) = A_3\psi_3(L_1),$$
$$\left[\frac{\partial}{\partial x}\psi_1(x) - \frac{1}{A_2^*}\frac{\partial}{\partial x}\psi_2(x) - \frac{1}{A_3^*}\frac{\partial}{\partial x}\psi_3(x)\right]|_{x=L_1} = \alpha\psi_1(L_1),\;\; A_2A_3 \neq 0.$$

The first boundary condition is matching condition and becomes continuity in special case $A_2 = A_3 = 1$, while second condition in this special case coincides with current conservation (α is assumed to be real) considered, for example, in the Refs. [16, 17]. We note that the eigenvalues of the linear Schrodinger equation on graphs can be found by solving a linear algebraic system following from the boundary conditions [16]. However, as we will see in the next section, for the stationary NLSE on graphs the boundary conditions lead to a system of transcendental equations and one should show the existence of its roots.

Consider the following time-independent NLSE with repulsive nonlinearity

$$-\psi_j'' + \beta_j|\psi_j|^2\psi_j = \lambda^2\psi_j,\; \beta_j > 0,\; j = 1,2,3. \tag{13.2}$$

given on the primary star graph presented in Fig. 13.1. The boundary conditions are given as (λ is real; $A_2 = \sqrt{\beta_2/\beta_1}$, $A_3 = \sqrt{\beta_3/\beta_1}$, $\alpha = 0$):

$$\psi_1(x)|_{x=0} = 0,\; \psi_2(x)|_{x=L_2} = \psi_3(x)|_{x=L_3} = 0, \tag{13.3}$$

$$\sqrt{\beta_1}\psi_1(L_1) = \sqrt{\beta_2}\psi_2(L_1) = \sqrt{\beta_3}\psi_3(L_1), \tag{13.4}$$

$$\left[\frac{1}{\sqrt{\beta_1}}\frac{\partial}{\partial x}\psi_1(x) - \frac{1}{\sqrt{\beta_2}}\frac{\partial}{\partial x}\psi_2(x) - \frac{1}{\sqrt{\beta_3}}\frac{\partial}{\partial x}\psi_3(x)\right]|_{x=L_1} = 0, \tag{13.5}$$

and the wave function is normalized as follows:

$$\sum_{j=1}^{3}\int_{b_j}|\psi_j(x)|^2dx = 1. \tag{13.6}$$

Dirichlet type boundary conditions are chosen in Eq. (13.3) due to their simplicity and their direct relevance to physical systems. It is easy to realize such conditions in physical systems than other ones.

The solution of Eq. (13.2) can be written as

$$\psi_j(x) = f_j(x)e^{i\gamma_j},\; j = 1,2,3, \tag{13.7}$$

where $\gamma_j = const$, $f_j(x)$ is a real function obeying the equation

$$-f_j'' + \beta_jf_j^3 = \lambda^2f_j. \tag{13.8}$$

The following relations can be obtained from the boundary conditions given by Eq. (13.4)

$$e^{i\gamma_1}\sqrt{\beta_1}f_1(L_1) = e^{i\gamma_2}\sqrt{\beta_2}f_2(L_1) = e^{i\gamma_3}\sqrt{\beta_3}f_3(L_1),$$

which lead to

$$\gamma_1 = \gamma_2 = \gamma_3 = \gamma,$$
$$\sqrt{\beta_1} f_1(L_1) = \sqrt{\beta_2} f_2(L_1) = \sqrt{\beta_3} f_3(L_1).$$

It is clear that the functions f_1, f_2, f_3 should obey Eqs. (13.3), (13.4), (13.5), and (13.6).

Exact solutions of Eq. (13.8) for finite interval and periodic boundary conditions can be found in the Refs. [9, 10]. Here we consider this problem for the graph boundary conditions given by Eq. (13.3). Solution of Eq. (13.8) satisfying these boundary conditions can be written as

$$\begin{aligned} f_1(x) &= B_1 sn\left(\alpha_1 x|k_1\right), \\ f_2(x) &= B_2 sn\left(\alpha_2 (x-L_2)|k_2\right), \\ f_3(x) &= B_3 sn\left(\alpha_3 (x-L_3)|k_3\right), \end{aligned}$$

where $sn(ax|k)$ are the Jacobian elliptic functions [34].

Inserting the last equation into Eq. (13.8) and comparing the coefficients of similar terms we have

$$B_j = \sigma_j \sqrt{\frac{2}{\beta_j}} \alpha_j k_j, \quad \lambda^2 = \alpha_j^2 \left(1 + k_j^2\right), \; j = 1,2,3, \tag{13.9}$$

where $\sigma_j = \pm 1 \; j = 1,2,3$.

Using Eqs. (13.4), (13.5), and (13.6) and the relations [34]

$$\int_a^b sn^2\left(\alpha(x-c)|k\right) dx = \frac{1}{k^2} \int_a^b \left[1 - dn^2\left(\alpha(x-c)|k\right)\right] dx$$

$$= \frac{1}{k^2}(b-a) - \frac{1}{\alpha k^2} E\left[am\left(\alpha(b-c)\right)|k\right] + \frac{1}{\alpha k^2} E\left[am\left(\alpha(a-c)\right)|k\right],$$

we obtain the system of transcendental equations with respect to α_j and k_j $(j = 1,2,3)$, which gives us the spectrum of the eigenvalues stationary NLSE on primary star graph:

$$\begin{aligned} \sqrt{\beta_1} B_1 sn\left(\alpha_1 L_1|k_1\right) = & \\ & = \sqrt{\beta_2} B_2 sn\left(\alpha_2 (L_1 - L_2)|k_2\right) \\ & = \sqrt{\beta_3} B_3 sn\left(\alpha_3 (L_1 - L_3)|k_3\right), \end{aligned} \tag{13.10}$$

$$\begin{aligned} \frac{B_1\alpha_1}{\sqrt{\beta_1}} cn\left(\alpha_1 L_1|k_1\right) dn\left(\alpha_1 L_1|k_1\right) - \frac{B_2\alpha_2}{\sqrt{\beta_2}} cn\left(\alpha_2 (L_1 - L_2)|k_2\right) dn\left(\alpha_2 (L_1 - L_2)|k_2\right) & \\ - \frac{B_3\alpha_3}{\sqrt{\beta_3}} cn\left(\alpha_3 (L_1 - L_3)|k_3\right) dn\left(\alpha_3 (L_1 - L_3)|k_3\right) = 0, & \end{aligned} \tag{13.11}$$

$$\frac{B_1^2}{k_1^2}L_1 + \frac{B_2^2}{k_2^2}(L_2-L_1)+\frac{B_3^2}{k_3^2}(L_3-L_1) =$$
$$= 1+\frac{B_1^2}{k_1^2\alpha_1}E\left[am(\alpha_1 L_1|k_1)|k_1\right]+\frac{B_2^2}{k_2^2\alpha_2}E\left[am(\alpha_2(L_2-L_1)|k_2)|k_2\right]+$$
$$+\frac{B_3^2}{k_3^2\alpha_3}E\left[am(\alpha_3(L_3-L_1)|k_3)|k_3\right]. \tag{13.12}$$

Here $E(\varphi|k)$ and $am(u|k)$ are the incomplete elliptic integral of the second kind and the Jacobi amplitude, respectively.

In general case this system can be solved using the different (e.g., Newton's or Krylov's method) iteration schemes. However, below we will show solvability of this system for two special cases.

Let

$$\frac{4n_1+1}{L_1}=\frac{4n_2+1}{L_2-L_1}=\frac{4n_3+1}{L_3-L_1},$$

where $n_1, n_2, n_3 \in \mathbf{N}\cup\{0\}$.

Choosing

$$\alpha_1=\frac{4n_1+1}{L_1}K(k_1),\ \alpha_2=\frac{4n_2+1}{L_2-L_1}K(k_2),\ \alpha_3=\frac{4n_3+1}{L_3-L_1}K(k_3),$$

we have

$$\alpha_1=\alpha_2=\alpha_3=\alpha,\ k_1=k_2=k_3=k,\ \sigma_1=1,\ \sigma_2=\sigma_3=-1.$$

Here $K(k)$ is the complete elliptic integral of the first kind.

It is clear that Eqs. (13.10) and (13.11) are valid under these conditions. Using Eq. (13.12) and the relations

$$am(u+2K(k)|k)=\pi+am(u|k),$$

$$E(n\pi\pm\varphi|k)=2nE(k)\pm E(\varphi|k),$$

we have

$$g(k)\equiv 2\frac{(4n_1+1)^2}{L_1^2}\left(\frac{L_1}{\beta_1}+\frac{L_2-L_1}{\beta_2}+\frac{L_3-L_1}{\beta_3}\right)K(k)\left(K(k)-E(k)\right)-1=0. \tag{13.13}$$

Solvability of Eq. (13.13) is equivalent to that of NLSE on primary star graph. Therefore we will prove solvability of this equation. Indeed, it follows from the relations

$$\lim_{k\to 0} g(k)=-1,\ \lim_{k\to 1} g(k)=+\infty,$$

and from the fact that $g(k)$ is a continuous function of k on the interval $(0;1)$, that Eq. (13.13) has a root.

Now we consider another special case given by the relations

$$\alpha_1 = \frac{(-1)^{n_1} p + 2n_1 K(k_1)}{L_1},\ \alpha_2 = \frac{(-1)^{n_2} p + 2n_2 K(k_2)}{L_2 - L_1},\ \alpha_3 = \frac{(-1)^{n_3} p + 2n_3 K(k_3)}{L_3 - L_1},$$

where $-K(k_j) \le p \le K(k_j)$, $n_j \in \mathbf{N}$, $j = 1,2,3$ and n_1, n_2, n_3 cannot be odd or even at the same time and show existence of the solution of the system given by Eqs. (13.10), (13.11), and (13.12). From Eqs. (13.9) and (13.10) we obtain

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha,\ k_1 = k_2 = k_3 = k,\ \sigma_1 = 1,\ \sigma_2 = \sigma_3 = -1.$$

From Eq. (13.11) we have

$$\frac{(-1)^{n_1}}{\beta_1} + \frac{(-1)^{n_2}}{\beta_2} + \frac{(-1)^{n_3}}{\beta_3} = 0.$$

Furthermore, it follows from the last equation and Eq. (13.12) that

$$g(k) \equiv 4\left(\frac{(-1)^{n_1} p + 2n_1 K(k_1)}{L_1}\right)\left(\frac{n_1}{\beta_1} + \frac{n_2}{\beta_2} + \frac{n_3}{\beta_3}\right)(K(k) - E(k)) - 1 = 0. \tag{13.14}$$

Therefore we have

$$\lim_{k\to 0} g(k) = -1,\ \lim_{k\to 1} g(k) = +\infty.$$

Since $g(k)$ is a continuous function of k on the interval $(0;1)$, it follows from the last relations that Eq. (13.14) has a root. Unlike the first special case, the second case describes primary star graph, with connected bonds.

13.3 Other Simplest Graphs

To extend the above approach to the case of other topologies, we consider a simplest topology, tree graph plotted in Fig. 13.2. Such an extension can be done using the same approach as that in the Ref. [28].

We seek the solution of Eq. (13.8) on the each bonds in the form

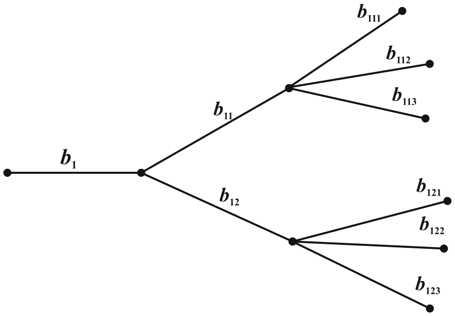
$$f_b = B_b sn\left(\alpha_b x + \delta_b \,|k_b\,\right),$$

where δ_b are parameters that can be determined by the given boundary conditions: $\delta_1 = 0,\ \delta_{1ij} = -\alpha_{1ij} L_{1ij}$.

Let us assume that the following relations are valid:

$$\frac{4n_1 + 1}{L_1} = \frac{4(n_{1i}^{(2)} - n_{1i}^{(1)})}{L_{1i} - L_1} = \frac{4n_{1ij} + 1}{L_{1ij} - L_{1i}},\ i = 1,2,\ j = 1,2,3$$

Fig. 13.2 Tree graph



where $n_1, n_{1i}, n_{1ij} \in \mathbf{N} \cup \{0\}, n_{1i}^{(2)} > n_{1i}^{(1)}$ and

$$\alpha_1 = \frac{4n_1+1}{L_1}K(k_1), \quad \alpha_{1i} = \frac{4(n_{1i}^{(2)} - n_{1i}^{(1)})}{L_{1i}-L_1}K(k_{1i}),$$

$$\alpha_{1ij} = \frac{4n_{1ij}+1}{L_{1ij}-L_{1i}}K(k_{1ij}), \;\; \delta_{1i} = \left(\frac{4(n_{1i}^{(1)}L_{1i} - n_{1i}^{(2)}L_1)}{L_{1i}-L_1} + 1\right)K(k_{1i}).$$

Then it is easy to show that these relations lead to equations

$$\alpha_1 = \alpha_{1i} = \alpha_{1ij} = \alpha, \; k_1 = k_{1i} = k_{1ij} = k, \; \sigma_1 = \sigma_{1i} = 1, \; \sigma_{1ij} = -1.$$

Therefore we have

$$g(k) \equiv 2\frac{(4n_1+1)^2}{L_1^2}\left\{\frac{L_1}{\beta_1} + \sum_{i=1}^{2}\left[\frac{L_{1i}-L_1}{\beta_{1i}} + \sum_{j=1}^{3}\frac{L_{1ij}-L_{1i}}{\beta_{1ij}}\right]\right\}K(k)$$
$$\times (K(k) - E(k)) - 1 = 0. \tag{13.15}$$

Solvability of the last equation is obvious.

Consider also the case given by the relations

$$\alpha_1 = \frac{-(-1)^{n_1}p_1 + 2n_1K(k_1)}{L_1},$$

$$\alpha_{1i} = \frac{(-1)^{n_{1i}^{(2)}}p_{1i} - (-1)^{n_{1i}^{(1)}}p_1 + 2(n_{1i}^{(2)} - n_{1i}^{(1)})K(k_{1i})}{L_{1i}-L_1},$$

$$\alpha_{1ij} = \frac{-(-1)^{n_{1ij}}p_{1i} + 2n_{1ij}K(k_{1ij})}{L_{1ij}-L_{1i}},$$

$$\delta_{1i} = \frac{(-1)^{n_{1i}^{(1)}}p_1L_{1i} - (-1)^{n_{1i}^{(2)}}p_{1i}L_1 + 2(n_{1i}^{(1)}L_{1i} - n_{1i}^{(2)}L_1)K(k_{1i})}{L_{1i}-L_1}.$$

where $-K(k_{1,1i}) < p_1 < K(k_{1,1i}), \; -K(k_{1i,1ij}) < p_{1i} < K(k_{1i,1ij})$.

One can obtain from the vertex conditions

$$\alpha_1 = \alpha_{1i} = \alpha_{1ij} = \alpha,\ k_1 = k_{1i} = k_{1ij} = k$$

and

$$\frac{(-1)^{n_1}}{\beta_1} - \sum_{i=1}^{2} \frac{(-1)^{n_{1i}^{(1)}}}{\beta_{1i}} = 0,\ \frac{(-1)^{n_{1i}^{(2)}}}{\beta_1} + \sum_{j=1}^{3} \frac{(-1)^{n_{1ij}}}{\beta_{1ij}} = 0.$$

It follows from the normalization condition that

$$g(k) \equiv\ 4\left(\frac{(-1)^{n_1} p + 2n_1 K(k)}{L_1}\right)\cdot \cdot\left(\frac{n_1}{\beta_1} + \sum_{i=1}^{2}\left[\frac{n_{1i}^{(2)} - n_{1i}^{(1)}}{\beta_{1i}} + \sum_{j=1}^{3}\frac{n_{1ij}}{\beta_{1ij}}\right]\right)(K(k) - E(k)) - 1 = 0.$$

Solvability of the last equation follows from the properties of the function $g(k)$. The same prescription can be repeated for loop graphs and combinations of loop and star graphs that give us similar treatment for these topologies.

13.4 Conclusions

We have studied time-independent NLSE with cubic nonlinearity for simplest networks and obtained explicit solutions for primary star and tree graphs are obtained by considering matching and flux conservation boundary conditions. Unlike the previous studies [28, 29, 32], the lengths of the bonds are considered as finite. Therefore our work can be considered as an extension of the earlier results by L.D. Carr et al. [9, 10, 12] to the case of networks. The method can be extended to other simplest graph topologies and their combinations.

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