Dynamics of inertial vortices in multicomponent Bose-Einstein condensates

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With use of the nonlinear Schrödinger (or Gross-Pitaevskii) equation with strong repulsive cubic nonlinearity, dynamics of multicomponent Bose-Einstein condensates (BECs) with a harmonic trap in two dimensions is investigated beyond the Thomas-Fermi regime. In the case when each component has a single vortex, we obtain an effective nonlinear dynamics for vortex cores (particles). The particles here acquire the inertia, in marked contrast to the standard theory of point vortices widely known in the usual hydrodynamics. The effective dynamics is equivalent to that of charged particles under a strong spring force and in the presence of Lorentz force with the uniform magnetic field. The interparticle (vortex-vortex) interaction is singularly repulsive and short ranged with its magnitude decreasing with increasing distance of the center of mass from the trapping center. "Chaos in the three-body problem" in the three-vortice system can be seen, which is not expected in the corresponding point vortices without inertia in two dimensions.

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I. INTRODUCTION

Recently, there has been much interest in theoretical and experimental studies on trapped atomic Bose-Einstein condensates (BECs) [1–3]. The superfluid property of atomic BECs arises from a dual aspect of waves and particles (i.e., matter waves), and is theoretically described by the macroscopic wave function. Because of the nonlinearity of the system caused by interaction between particles, the macroscopic wave function can take a form of various solitons such as bright, dark, gray, and vortex solitons, and these solitons are experimentally observed in BECs.

As for bright solitons, Martin *et al.* [4] theoretically predicted that the particlelike behavior of three bright solitons in a one-dimensional ^{87}Rb BEC was nonintegrable and showed its change from regular motions to chaos. Pérez-García *et al.* [5] applied a variational method to dynamics of bright solitons in a two-dimensional (2D) BEC and showed that the center of mass of each soliton obeys Newtonian dynamics and Ehrenfest's theorem is valid if the phase of the BEC wave function will be suitably chosen.

As more interesting systems, we can consider solitons in multicomponent BECs which consist of different kinds of atoms or the same kind of atoms having different spin and that have been experimentally realized. In multicomponent BECs, there is not only intracomponent particle interaction but also intercomponent particle interaction which is another origin of nonlinearity, so we expect novel soliton dynamics which is not seen in single-component BECs. Yamasaki *et al.* [6] developed a variational method to describe bright soliton dynamics in 2D multicomponent BECs, and proposed a model of conservative chaos. In 2D and three-dimensional (3D) systems, however, bright solitons are unstable unless intracomponent interaction oscillates between attraction and repulsion or coexisting intracomponent quintic (three-body) interaction is strong enough.

On the other hand, topological vortices known as quantized vortices (i.e., topological defects of the macroscopic wave function can be stable in two dimensions). Vortices in single-component [7–9] and multicomponent [10–12] BECs have been realized experimentally, making a good candidate to study the dynamics of vortices in 2D and 3D BECs. Most of the theoretical studies, however, are limited to the Thomas-Fermi regime (TFR) in a single-component BEC [9,13–15]. While dynamics of the macroscopic wave function of BEC is described by Gross-Pitaevskii equation (GPE) in Eq. (2) below, TFR suppresses the kinetic energy part in GPE in the lowest approximation and therefore the lowest-order wave function cannot have a healing length which is a hallmark of the vortex core. It is highly desirable to construct the effective theory of vortices beyond the TFR.

In this paper, we consider the vortices in 2D multicomponent BECs beyond the Thomas-Fermi regime. To consider the effective dynamics of pointlike vortices, we extract some degrees of freedom by using the variational approach, and derive an effective dynamics with finite degrees of freedom. For the case of a single-component BEC without a trap, it is well known that the effective Hamiltonian for many vortex systems has a standard form [16–18],

$$H \sim \sum_{j>i} n_i n_j \ln r_{ij}, \tag{1}$$

in the limit of infinitesimal vortex cores. Here n_i is the winding number of the ith vortex, and $r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$ is the distance between cores of ith and jth vortices. Equation (1) shows that there is no momentum degree of freedom, and coordinates x_i and y_i formally conjugate each other. On the other hand, in the case of multicomponent BECs in a trap with each component having a single vortex, we shall see vortices acquire momentum degrees of freedom or inertia. A very preliminary idea of the present work was reported in the conference proceedings [19].

This paper is organized as follows. In Sec. II, starting from the multicomponent GPE, we apply the variational method with use of vortex solitons in the Padé approximation, and derive the effective Hamiltonian for vortices. Confining to the case of two and three vortices, we numerically investigate the detailed dynamics of vortices in Sec. III. There we see a chaotic behavior of the system consisting of three vortices. Section IV is devoted to conclusions and discussions.

II. EFFECTIVE NONLINEAR DYNAMICS GENERATED BY THE MULTICOMPONENT GPE

In this section, we consider the trapped multicomponent GPE with vortices and extract some degrees of freedom of vortex soliton by using a variational technique.

BEC at zero temperature is described by the GPE. We shall consider a 2D system of trapped n-component macroscopic wave function $\Phi_1(t,x,y),\Phi_2(t,x,y),\ldots,\Phi_n(t,x,y)$ satisfying the equations,

$$i\frac{\partial}{\partial t}\Phi_i(t,x,y) = \left[-\nabla^2 + V(x,y) + g_{ii}|\Phi_i(t,x,y)|^2 + \sum_{j\neq i} g_{ij}|\Phi_j(t,x,y)|^2 \right]\Phi_i(t,x,y), \quad (2)$$

for $i,j = 1,\ldots,n$. Here the normalization condition for each component of wave functions is defined by $\int |\Phi_i(t,x,y)|^2 dxdy = 1$ after a proper rescaling of Φ_i by the particle number N common to all components. The effect of trapping is expressed by $V(x,y) = (x^2+y^2)$. Equation (2) is expressed with use of scaled variables: using the confining length $l = \sqrt{\frac{\hbar}{m\omega}}$ and oscillation period $\tau = \omega^{-1}$, space coordinates are scaled by l, time by 2τ, wave function by $\frac{1}{l}$, and nonlinearity by $\frac{\hbar^2}{2m}$. The nonlinearity coefficients, $g_{ij} \equiv 8\pi N a_{ij}/l$ with a_{ij} the scattering length for binary collisions, are assumed to be positive and much larger than unity. g_{ii} and g_{ij} with $i \neq j$ stand for intracomponent and intercomponent interactions, respectively. We shall choose $g_{ii} = g_1(\gg 1)$ for all i and $g_{ij} = g_2(\gg 1)$ for $i \neq j$.

In the absence of the intercomponent interaction, each component has stationary states of a vortex. So, we consider the case in which each component has one vortex and vortices interact with each other through the intercomponent interaction. Our goal is to derive from Eq. (2) the evolution equation for the collective coordinates of trial vortex functions (TVFs). The collective coordinates for a vortex are phase variables besides the coordinates of a vortex core. We shall use TVF beyond the Thomas-Fermi regime, by incorporating the effect of a kinetic energy in GPE in Eq. (2): As for the amplitude of TVF, we choose a vortex function based on the Padé approximation [20,21] which is regularized due to a trap. As for its phase, we Taylor expand the phase with respect to space coordinates around the vortex core. Then TVF with winding number $n_i = \pm 1$ is given by

$$\Phi_i(t,x,y) \equiv f_i(t,x,y)\exp[i\phi_i(t,x,y)] = N\exp\left[-\frac{x^2+y^2}{2\Delta}\right]\sqrt{\frac{(x-x_i)^2+(y-y_i)^2}{2\xi^2+(x-x_i)^2+(y-y_i)^2}} \times \exp\left[i\left[n_i\tan^{-1}\left(\frac{y-y_i}{x-x_i}\right)+\alpha_i(x-x_i)+\beta_i(y-y_i)\right]\right], \quad (3)$$

with the normalization factor $N = \frac{1}{\sqrt{\pi\Delta - 2\pi e^{\frac{2\xi^2}{\Delta}}\xi^2\Gamma(0,\frac{2\xi^2}{\Delta})}}$. Here $\Gamma(0,z) \equiv \int_z^\infty t^{-1}e^{-t}dt$ is the incomplete gamma function of the second kind, whose expansion with respect to z is given in Eq. (A10) in Appendix A.

The collective coordinates are locations of the core (x_i,y_i) and the first-order coefficients (α_i,β_i) of Taylor expansion of the phase $\phi_i(t,x,y)$ with respect to $(x-x_i,y-y_i)$. Δ in the Gaussian amplitude factor reflects a trap in Eq. (3). ξ is the healing length related to vortex core size. The condition to minimize the energy $E = \int dxdy(|\nabla\Phi_i|^2 + V(x,y)|\Phi_i|^2 + \frac{g_1}{2}|\Phi_i|^4)$ for the individual static component in Eq. (3) centered at the origin, leads to $\Delta \cong \sqrt{\frac{3}{2} - (\gamma+1)n_i^2 + \frac{g_1}{4\pi}}$ and $\xi \cong \frac{|n_i|\pi^{1/4}}{\sqrt{2+\gamma}} g_1^{-1/4}$, respectively, where $\gamma(=0.57721)$ is Euler constant. We shall use these values for Δ and ξ in this paper.

The form in Eq. (3), which is a product of the vortex solution in the absence of a harmonic trap and Gaussian factor due to the trap, gives a suitable TVF for a vortex under the strong nonlinearity. A different form with use of eigenstates (with nonzero angular momenta) under the 2D harmonic trap [22–24] has no small healing length, results in the intervortices force growing with intervortices distance, etc., and cannot be suitable as TVF under the strong nonlinearity.

First of all we note: GPE in Eq. (2) can be derived from the variational principle that minimizes the action obtained from Lagrangian density $\mathcal{L}$ for field variables,

$$-\mathcal{L} = \sum_i \left[\frac{i}{2}(\Phi_i\dot{\Phi}_i^* - \Phi_i^*\dot{\Phi}_i) + |\nabla\Phi_i|^2 + (x^2+y^2)|\Phi_i|^2 + \frac{g_1}{2}|\Phi_i|^4\right] + \sum_{j>i} g_2|\Phi_i|^2|\Phi_j|^2. \quad (4)$$

In fact, Eq. (2) is obtained from the Lagrange equation:

$$\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\dot{\Phi}_i^*} - \frac{\partial\mathcal{L}}{\partial\Phi_i^*} + \nabla\frac{\partial\mathcal{L}}{\partial\nabla\Phi_i^*} = 0. \quad (5)$$

We now insert TVF in Eq. (3) into Eq. (4). Noting (x_i,y_i) and (α_i,β_i) as time-dependent variables, Eq. (4) becomes

$$-\mathcal{L} = \sum_i \left[\left(\dot{x}_i\frac{\partial\phi_i}{\partial x_i} + \dot{y}_i\frac{\partial\phi_i}{\partial y_i} + \dot{\alpha}_i\frac{\partial\phi_i}{\partial\alpha_i} + \dot{\beta}_i\frac{\partial\phi_i}{\partial\beta_i}\right)f_i^2 + \left(\frac{\partial f_i}{\partial x}\right)^2 + \left(\frac{\partial f_i}{\partial y}\right)^2 + \left(\left(\frac{\partial\phi_i}{\partial x}\right)^2 + \left(\frac{\partial\phi_i}{\partial y}\right)^2\right)f_i^2 + \frac{g_1}{2}f_i^4 + (x^2+y^2)f_i^2\right] + g_2\sum_{j>i} f_i^2 f_j^2. \quad (6)$$

TABLE I. Expressions for coefficients in Eq. (8) (Taylor expansions with respect to ξ^2/Δ).

Coefficients	Expressions
d_1	$1+\frac{2\gamma\xi^2}{\Delta}+\frac{2\xi^2}{\Delta}\ln\frac{2\xi^2}{\Delta}$
d_2	$\frac{1}{2}-\frac{\xi^2}{\Delta}$
d_3	$-\frac{\gamma}{2}-\frac{1}{2}\ln\frac{2\xi^2}{\Delta}+\frac{\xi^2}{\Delta}-\frac{\gamma\xi^2}{\Delta}-\frac{\xi^2}{\Delta}\ln\frac{2\xi^2}{\Delta}$
d_4	$\frac{3}{2}+\frac{\xi^2}{\Delta}+\frac{4\gamma\xi^2}{\Delta}+\frac{4\xi^2}{\Delta}\ln\frac{2\xi^2}{\Delta}$
d_5	$\frac{1}{2}+\frac{\gamma\xi^2}{\Delta}+\frac{\xi^2}{\Delta}\ln\frac{2\xi^2}{\Delta}$
d_6	$1+\frac{5\xi^2}{\Delta}+\frac{6\gamma\xi^2}{\Delta}+\frac{6\xi^2}{\Delta}\ln\frac{2\xi^2}{\Delta}$
d_7	$\frac{\xi^2}{\Delta}+\frac{2\gamma\xi^2}{\Delta}+\frac{2\xi^2}{\Delta}\ln\frac{4\xi^2}{\Delta}$
d_8	$\Delta-2\xi^2$
d_9	$1+\frac{2\xi^2}{\Delta}+\frac{2\gamma\xi^2}{\Delta}+\frac{2\xi^2}{\Delta}\ln\frac{2\xi^2}{\Delta}$

By integrating $\mathcal{L}$ over space coordinates (x,y), we obtain the effective Lagrangian L for the collective coordinates:

$$L=\iint dx\,dy\,\mathcal{L}. \tag{7}$$

In the limit of $\xi^2/\Delta \ll 1$, L is expressed by

$$\begin{aligned}-L=&\left(\frac{N}{N_0}\right)^2\sum_i\Bigg[-\frac{n_i}{\Delta}(\dot{x}_iy_i-\dot{y}_ix_i)\,e^{-\frac{l_i^2}{\Delta}}\left(d_1+\frac{d_2l_i^2}{\Delta}\right)\\&+\left(\alpha_i^2+\beta_i^2-(\alpha_i\dot{x}_i+\beta_i\dot{y}_i)\right)d_1\\&-(\dot{\alpha}_ix_i+\dot{\beta}_iy_i)\left(1-\frac{2\xi^2}{\Delta}\left(1-\frac{l_i^2}{2\Delta}\right)\right)\\&+\frac{1}{\Delta}e^{-\frac{l_i^2}{\Delta}}\left(2n_i^2d_3+d_4+2n_i^2\frac{l_i^2d_5}{\Delta}+d_6\frac{l_i^2}{\Delta}\right)\\&+2\frac{n_i}{\Delta}(y_i\alpha_i-x_i\beta_i)\,e^{-\frac{l_i^2}{\Delta}}\left(d_1+\frac{l_i^2}{\Delta}d_2\right)\\&+e^{-\frac{l_i^2}{\Delta}}(d_8+l_i^2d_9)+\left(\frac{N}{N_0}\right)^2\frac{g_1}{\pi\Delta}\left(\frac{1}{4}+d_7e^{-\frac{2l_i^2}{\Delta}}\right)\Bigg]\\&+\sum_{j>i}U\left(r_{ij},l_G^{ij}\right).\end{aligned} \tag{8}$$

Here, $l_i=\sqrt{x_i^2+y_i^2}$ and $N_0=\frac{1}{\sqrt{\pi\Delta}}$. Expressions (Taylor expansions with respect to ξ^2/Δ) for coefficients $d_1\sim d_9$ are listed in Table I and the derivation of typical coefficients is described in Appendix A. The expression for the intervortice interaction $U(r_{ij},l_G^{ij})$, which is a function of intervortice distance $r_{ij}=\sqrt{x_{ij}^2+y_{ij}^2}$ and distance of the center of masses from the origin $l_G^{ij}=\sqrt{(x_G^{ij})^2+(y_G^{ij})^2}$, will be given in Appendix B.

Lagrange equations of motion for the phase variables α_i and β_i,

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\alpha}_i}\right)-\frac{\partial L}{\partial\alpha_i}=0,\\\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\beta}_i}\right)-\frac{\partial L}{\partial\beta_i}=0,\end{aligned} \tag{9}$$

lead to

$$\alpha_i\simeq B_1\dot{x}_i-n_iB_2y_i,\quad \beta_i\simeq B_1\dot{y}_i+n_iB_2x_i. \tag{10}$$

Equation (10) shows that (α_i,β_i) correspond to generalized momentum conjugate to (x_i,y_i) under the vector potential. Then the equation of motion for (x_i,y_i),

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}_i}\right)-\frac{\partial L}{\partial x_i}=0, \tag{11}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{y}_i}\right)-\frac{\partial L}{\partial y_i}=0, \tag{12}$$

combined with Eq. (10), gives

$$\begin{aligned}\ddot{x}_i&\simeq n_iB_3\dot{y}_i-B_4x_i-B_5\sum_{j\neq i}\frac{\partial U(x_{ij},y_{ij})}{\partial x_i},\\\ddot{y}_i&\simeq -n_iB_3\dot{x}_i-B_4y_i-B_5\sum_{j\neq i}\frac{\partial U(x_{ij},y_{ij})}{\partial y_i}.\end{aligned} \tag{13}$$

Coefficients $B_1\sim B_5$ are given by

$$\begin{aligned}B_1&=\frac{d_1-1+\frac{2\xi^2}{\Delta}}{2d_1}\sim\frac{(\gamma+1)\xi^2}{\Delta},\\B_2&=\frac{1}{\Delta},\\B_3&=-\frac{4d_1^2}{\Delta\left(d_1-1+\frac{2\xi^2}{\Delta}\right)^2}\sim-\frac{\Delta}{(\gamma+1)^2\xi^4},\\B_4&=\frac{4d_1d_9}{\left(d_1-1+\frac{2\xi^2}{\Delta}\right)^2}\sim\frac{\Delta^2}{(\gamma+1)^2\xi^4},\\B_5&=\left(\frac{N}{N_0}\right)^2\frac{2d_1}{\left(d_1-1+\frac{2\xi^2}{\Delta}\right)^2}\sim\frac{\Delta^2}{2(\gamma+1)^2\xi^4}.\end{aligned} \tag{14}$$

Equations (10) and (13) are valid aside from a multiplicative global factor $[1+O(\frac{l_i^2}{\Delta})]$. The smallness of $\frac{l_i^2}{\Delta}$ will be justified *a posteriori*. Equation (13) shows that dynamics of coordinates (x_i,y_i) is very similar to charged particles with charges $n_i=\pm1$ under a strong spring force with a force constant $B_4=O(g_1^2)$ and in the presence of Lorentz force with the magnetic field $\mathbf{B}=(0,0,B_3)$. One should note that the spring force here has nothing to do with that of the original harmonic potential $V(x,y)$ in Eq. (2). The Hamiltonian corresponding to Eq. (13) can be given by

$$H=\sum_i\left[\frac{1}{2m}[\mathbf{p}_i-n_i\mathbf{A}_i]^2+W_i+\sum_{j>i}\tilde{U}\left(r_{ij},l_G^{ij}\right)\right], \tag{15}$$

with use of the momentum $\mathbf{p}_i\equiv m\dot{\mathbf{r}}+n_i\mathbf{A}_i$, unit mass $m=1$, the vector potential $\mathbf{A}_i=\frac{B_3}{2}(-y_i,x_i)$, and the scalar potential $W_i=\frac{1}{2}B_4(x_i^2+y_i^2)$.

All values $B_1\sim B_5$ depend on the strength of interaction g_1 which is tunable by Feshbach resonance. In the effective particle dynamics described by Eqs. (13) and (15), the spring constant B_4 is large enough to guarantee particles with unit mass to be confined in the neighborhood of the origin (i.e., the trapping center). This finding justifies $\frac{l_i^2}{\Delta}\ll1$ and will also be utilized to obtain the intervortice interaction in Appendix B.

The scaled intervortice interaction $\tilde{U}(r_{ij},l_G^{ij})$ is given with use of Eq. (B10) as

$$\tilde{U}(r_{ij},l_G^{ij}) = B_5 U_4(r_{ij},l_G^{ij}) \cong \frac{G}{r_{ij}^2}\exp\left(-\frac{r_{ij}^2+4(l_G^{ij})^2}{2\Delta}\right), \quad (16)$$

with the coupling constant,

$$G = \frac{4g_2}{(\gamma+1)^2\pi^{1/2}}. \quad (17)$$

Therefore the intervortice interaction is singularly repulsive and short ranged with respect to r_{ij} with its magnitude decreasing with increasing l_G^{ij}.

Compared to Eq. (1), it is clear that the system has momentum degrees of freedom, and vortices have a behavior of particle with inertia rather than that of vortex point without inertia widely used in the conventional theory of hydrodynamics [25–29]. This is one of the main assertions of the present paper. The inertia of vortex appears already in the single-component BEC with a trap.

In closing this section we should comment that we also attempted to apply the collective coordinate method using a Laguerre-type trial function which is a good candidate for TVF in the case of a weak nonlinearity [22–24]. However, such TVF in the case of a strong nonlinearity proved to result in (1) a time-dependent mass for each vortex and (2) the intervortex interaction growing with increasing intervortex distance. Hence this TVF was not suitable to describe a dynamics of vortices in BECs with a strong nonlinearity. On the other hand, the effective vortex dynamics in the Thomas-Fermi regime [9,13–15], which suppresses the kinetic energy in constructing TVF, leads to neither nonzero inertia nor Lorentz force.

III. DYNAMICS OF TWO AND THREE VORTICES WITH INERTIA

We shall now focus on the system of two vortices with equal winding numbers, and see how the trajectory generated by effective particle dynamics in Eqs. (13) and (15) well mimics the orbit of the singular points of wave vortices calculated by using GPE in Eq. (2). We shall then move to the system of three vortices with equal winding numbers, and find that chaos appears even in the three-vortex system. This feature is different from that of the point vortice system in a single-component BEC in which chaos can appear in the case of more than three vortices.

A. Dynamics of two vortices

For two vortices with the same winding numbers $n_1 = n_2 = 1$, Hamiltonian (15) can be rewritten as

$$H = \frac{1}{2}\left(p_{x,T}^2 + p_{y,T}^2 + p_{x,R}^2 + p_{y,R}^2\right) - \frac{B_3}{2}(p_{x,T}y_T - p_{y,T}x_T + p_{x,R}y_R - p_{y,R}x_R) + \left(\frac{B_3^2}{8}+\frac{B_4}{2}\right)\left(x_T^2+y_T^2+x_R^2+y_R^2\right) + \frac{G}{2(x_R^2+y_R^2)}\exp\left(-\frac{x_R^2+y_R^2+x_T^2+y_T^2}{\Delta}\right), \quad (18)$$

where (x_T,y_T) and (x_R,y_R) play the role of the center-of-mass and relative coordinates, respectively, and $(p_{x,T},p_{y,T})$ and $(p_{x,R},p_{y,R})$ are their canonical-conjugate variables. To be explicit,

$$(x_T,y_T) = \frac{1}{\sqrt{2}}(x_1+x_2, y_1+y_2),$$
$$(x_R,y_R) = \frac{1}{\sqrt{2}}(x_1-x_2, y_1-y_2),$$
$$(p_{x,T},p_{y,T}) = \frac{1}{\sqrt{2}}(p_{x,1}+p_{x,2}, p_{y,1}+p_{y,2}),$$
$$(p_{x,R},p_{y,R}) = \frac{1}{\sqrt{2}}(p_{x,1}-p_{x,2}, p_{y,1}-p_{y,2}). \quad (19)$$

Hamiltonian (15) cannot be reduced to two independent two degree-of-freedom subsystems (center-of-mass system and relative-coordinate system) because the intervortex interaction depends not only on r_{ij} but also on l_G^{ij}. However, we see mostly KAM tori in this system. As shown in Fig. 1, we find that the trajectory of (x_R,y_R) generated by effective two-particle dynamics well mimics the corresponding orbit obtained by a

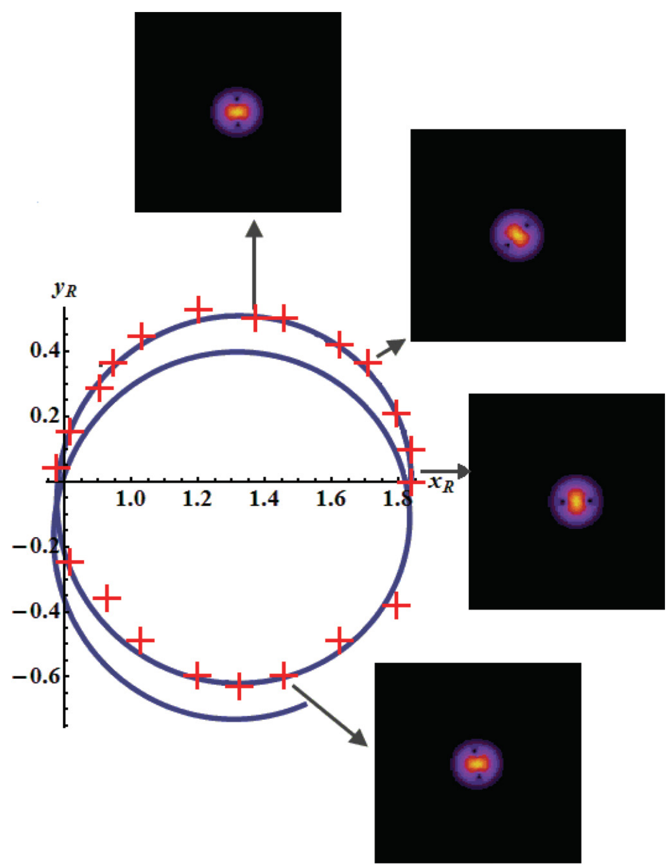

FIG. 1. (Color online) Two-vortice dynamics with identical winding numbers ($n_1 = n_2 = 1$). $g_1 = g_2 = 100$. Initial values are as follows: $x_1(0) = y_1(0) = -x_2(0) = -y_2(0) = \frac{1}{\sqrt{2}}$; $\dot{x}_1(0) = \dot{x}_2(0) = 0$; $\dot{y}_1(0) = -\dot{y}_2(0) = 1.1$. Solid line is a trajectory for relative coordinates of a pair of particles calculated by iterating the canonical equations for the Hamiltonian (18); crosses are corresponding results for a pair of singular points of interacting vortices constructed from the numerical iteration of GPE in Eq. (2); four subpanels are wave-function patterns for interacting vortices.

pair of singular points calculated by using GPE in Eq. (2). This fact justifies the validity of our trial function in Eq. (3) and the resultant equation of motion for collective coordinates in Eqs. (13) and (15).

B. Dynamics of three vortices

Encouraged by the effectiveness of the collective coordinate method in the case of two vortices, we proceed to the dynamics of three vortices with the identical winding numbers $n_1 = n_2 = n_3 = 1$, whose Hamiltonian (15) becomes

$$\begin{aligned} H = {} & \frac{1}{2}\left(p_{xC}^2 + p_{yC}^2 + p_{xT}^2 + p_{yT}^2 + p_{xR}^2 + p_{yR}^2\right) \\ & - \frac{B_3}{2}(x_C p_{yC} + x_R p_{yR} + x_T p_{yT} \\ & - y_C p_{xC} - y_R p_{xR} - y_T p_{xT}) \\ & + \left(\frac{B_3^2}{8} + \frac{B_4}{2}\right)\left(x_C^2 + x_R^2 + x_T^2 + y_C^2 + y_R^2 + y_T^2\right) \\ & + \sum_{(i,j)=(1,2),(2,3),(3,1)} \frac{G}{(x_i - x_j)^2 + (y_i - y_j)^2} \\ & \times \exp\left(-\frac{x_i^2 + y_i^2 + x_j^2 + y_j^2}{\Delta}\right), \end{aligned} \tag{20}$$

where Jacobi coordinates (x_T, y_T, etc) are defined by

$$\begin{aligned} (x_T, y_T) &= \frac{1}{\sqrt{3}}(x_1 + x_2 + x_3, y_1 + y_2 + y_3), \\ (x_C, y_C) &= \frac{1}{\sqrt{2}}(x_1 - x_3, y_1 - y_3), \\ (x_R, y_R) &= \frac{1}{\sqrt{6}}(x_1 + x_3 - 2x_2, y_1 + y_3 - 2y_2), \\ (p_{xR}, p_{yR}) &= \frac{1}{\sqrt{6}}(p_{x1} + p_{x3} - 2p_{x2}, p_{y1} + p_{y3} - 2p_{y2}), \\ (p_{xT}, p_{yT}) &= \frac{1}{\sqrt{3}}(p_{x1} + p_{x2} + p_{x3}, p_{y1} + p_{y2} + p_{y3}), \\ (p_{xC}, p_{yC}) &= \frac{1}{\sqrt{2}}(p_{x2} - p_{x1}, p_{y2} - p_{y1}), \end{aligned} \tag{21}$$

which conserve the canonical structure $(\{x_T, p_{xT}\} = \frac{1}{3}\sum_{j=1}^{3}\{x_j, p_{xj}\} = 1$, etc.). Here (x_T, y_T), (x_C, y_C), and (x_R, y_R) represent the center of mass of all three components, the relative displacement, and the bisector of the vertex (x_2, y_2), respectively. In marked contrast to the massless three vortex system in two dimensions which is integrable, all six degrees of freedom are coupled and the number of independent constants of motion is two (energy and z component of angular momentum). Then Poincaré-Bendixon's theorem guarantees the nonintegrability and chaos of the three-inertial-vortex system [30].

We can construct from (20) the canonical equations of motion for $x_T, y_T, x_C, y_C, x_R, y_R$ and their canonical-conjugate variables, which are solved numerically. Poincaré cross section and power spectra for the kinetic energy $E_k(t)$ in Fig. 2 give a clear evidence of high-dimensional chaos. Because of the short-range nature of the interaction, we see the emergence of chaos in low-lying energy regions where vortices often meet each other.

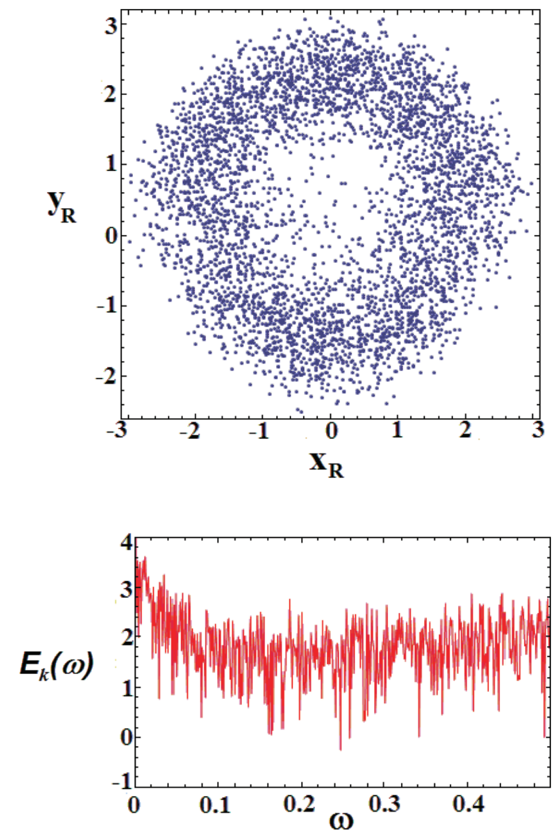

FIG. 2. (Color online) Three-vortice dynamics with identical winding numbers $(n_1 = n_2 = n_3 = 1)$. $g_1 = g_2 = 100$. Initial values are as follows: $x_R(0) = x_C(0) = x_T(0) = 1$, $y_R(0) = -1$, $y_C(0) = y_T(0) = 1$; $p_{x,R}(0) = p_{x,C}(0) = p_{x,T}(0) = p_{y,R}(0) = p_{y,C}(0) = p_{y,T}(0) = 1$. (a) Poincaré cross section; (b) Power spectra for kinetic energy $E_k(t)$.

IV. CONCLUSION

We explored vortex dynamics in the two-dimensional multicomponent BEC in the harmonic trap in the case that each component has a single vortex. The investigation beyond the Thomas-Fermi regime is made on the nonlinear Schrödinger equation with strong repulsive cubic nonlinearity. With use of a trial vortex function based on the Padé approximation which is regularized due to a trap, we applied a collective-coordinates method, obtaining an effective nonlinear dynamics for vortex cores (particles), which is equivalent to charged particles with inertia under a strong spring force and in the presence of Lorentz force with the uniform magnetic field. The interparticle interaction is singularly repulsive and short ranged with its magnitude decreasing with increasing distance of the center of mass from the trapping center. The most important finding is the nonzero inertia of vortices, which is not present in the conventional theory of point vortices widely used in the usual hydrodynamics [25–29]. The system of three vortices with inertia can be chaotic, in contrast to the corresponding case of point vortices without inertia.

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APPENDIX A: CALCULATION OF TYPICAL INTEGRALS

We shall calculate some integrals used in this paper.

1. Integral $A \equiv \iint dxdy f_i^2 \left(\dot{x}_i \frac{\partial\phi_i}{\partial x_i} + \dot{y}_i \frac{\partial\phi_i}{\partial y_i}\right)$

Substituting Eq. (3) into the above integrand, we find

$$\begin{aligned} A = &\frac{1}{\pi\Delta}\left(\frac{N}{N_0}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy e^{-\frac{x^2+y^2}{\Delta}} \\ &\times\left[n_i \frac{\dot{x}_i(y-y_i)-\dot{y}_i(x-x_i)}{2\xi^2+(x-x_i)^2+(y-y_i)^2}\right. \\ &\left.-(\alpha_i\dot{x}_i+\beta_i\dot{y}_i)\frac{(x-x_i)^2+(y-y_i)^2}{2\xi^2+(x-x_i)^2+(y-y_i)^2}\right], \end{aligned} \tag{A1}$$

where $N_0 = \frac{1}{\sqrt{\pi\Delta}}$ and N is the normalization constant defined below Eq. (3). Equation (A1) is a sum of n_i-dependent term (A_1) and (α_i,β_i)-dependent term (A_2). Below we shall concentrate on A_1 because A_2 is easily calculable. Using polar coordinates as

$$\begin{aligned} x-x_i &= \rho\cos\theta, \quad x_i = l_i\cos\phi_i, \\ y-y_i &= \rho\sin\theta, \quad y_i = l_i\sin\phi_i, \\ dxdy &= \rho d\rho d\theta, \end{aligned} \tag{A2}$$

we rewrite the integral A_1 as

$$\begin{aligned} A_1 = &\frac{n_i}{\pi\Delta}\left(\frac{N}{N_0}\right)^2 \int_0^{\infty} d\rho e^{-\frac{\rho^2+l_i^2}{\Delta}} \frac{\rho^2}{2\xi^2+\rho^2} \\ &\times\int_0^{2\pi} d\theta(\dot{x}_i\sin\theta-\dot{y}_i\cos\theta)e^{-\frac{2\rho l_i}{\Delta}\cos(\theta-\phi_i)}. \end{aligned} \tag{A3}$$

The θ integration gives

$$\begin{aligned} &\int_0^{2\pi} d\theta(\dot{x}_i\sin\theta-\dot{y}_i\cos\theta)e^{-\frac{2\rho l_i}{\Delta}\cos(\theta-\phi_i)} \\ &= \int_0^{2\pi} d\tilde{\theta}[(\dot{x}\cos\phi_i+\dot{y}\sin\phi_i)\sin\tilde{\theta} \\ &\quad +(\dot{x}\sin\phi_i-\dot{y}\cos\phi_i)\cos\tilde{\theta}]e^{-\frac{2\rho l_i}{\Delta}\cos\tilde{\theta}} \\ &= -\pi(\dot{x}_i y_i-\dot{y}_i x_i)\left(\frac{2\rho}{\Delta}+\frac{\rho^3 l_i^2}{\Delta^3}\right). \end{aligned} \tag{A4}$$

Here we took $\tilde{\theta} = \theta - \phi_i$ and used the formulas like

$$\int_0^{2\pi} d\tilde{\theta}\cos\tilde{\theta} e^{-z\cos\tilde{\theta}} = -2\pi I_1(z), \tag{A5}$$

where $I_1(z)$ is the modified Bessel function, which is expanded as

$$I_1(z) = \frac{z}{2} + \frac{z^3}{16} + O(z^5). \tag{A6}$$

As for ρ integration, we use identities like

$$\frac{\rho^3}{2\xi^2+\rho^2} = \rho - \frac{2\xi^2\rho}{2\xi^2+\rho^2}, \tag{A7}$$

and then apply the formulas,

$$\int_0^{\infty} d\rho\rho^n e^{-\frac{\rho^2}{\Delta}} = \frac{1}{2}\Delta^{\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right), \tag{A8}$$

and

$$\begin{aligned} \int_0^{\infty} d\rho e^{-\frac{\rho^2}{\Delta}} \frac{\rho}{2\xi^2+\rho^2} &= \frac{1}{2}e^{\frac{2\xi^2}{\Delta}}\Gamma\left(0,\frac{2\xi^2}{\Delta}\right) \\ &= \frac{1}{2}e^{\frac{2\xi^2}{\Delta}}\left(-\gamma - \ln\frac{2\xi^2}{\Delta} + \frac{2\xi^2}{\Delta} - \left(\frac{\xi^2}{\Delta}\right)^2\right). \end{aligned} \tag{A9}$$

$\Gamma(0,z) \equiv \int_z^{\infty} t^{-1}e^{-t}dt$ above is the incomplete gamma function of the second kind and can be expanded as

$$\Gamma(0,z) = -\gamma - \ln z + z - \frac{z^2}{4} + \frac{z^3}{18} + O(z^4). \tag{A10}$$

The final result is

$$A_1 = -\left(\frac{N}{N_0}\right)^2 \frac{n_i}{\Delta}(\dot{x}_i y_i - \dot{y}_i x_i)e^{-\frac{l_i^2}{\Delta}}\left(d_1 + \frac{d_2 l_i^2}{\Delta}\right), \tag{A11}$$

where d_1 and d_2 are listed in Table I. Equation (A11), combined with

$$A_2 = -\left(\frac{N}{N_0}\right)^2 (\alpha_i\dot{x}_i + \beta_i\dot{y}_i)d_1, \tag{A12}$$

gives rise to the calculated result for A in Eq. (A1).

2. Integral $B \equiv \frac{g_1}{2}\iint f_i^4 dxdy$

$$\begin{aligned} B = &\frac{g_1}{2\pi^2\Delta^2}\left(\frac{N}{N_0}\right)^4 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy e^{-\frac{2(x^2+y^2)}{\Delta}} \\ &\times\left(\frac{(x-x_i)^2+(y-y_i)^2}{2\xi^2+(x-x_i)^2+(y-y_i)^2}\right)^2. \end{aligned} \tag{A13}$$

Using polar coordinates in (A2), we rewrite the integral as

$$\begin{aligned} B = &\frac{g_1}{2\pi^2\Delta^2}\left(\frac{N}{N_0}\right)^4 \int_0^{\infty}\rho d\rho e^{-\frac{2(\rho^2+l_i^2)}{\Delta}} \\ &\times\left(\frac{\rho^2}{2\xi^2+\rho^2}\right)^2 \int_0^{2\pi} d\theta e^{-\frac{4\rho l_i}{\Delta}\cos(\theta-\phi_i)}. \end{aligned} \tag{A14}$$

First, we carry out the θ integration. Note that

$$\int_0^{2\pi} d\theta e^{-z\cos\theta} = 2\pi I_0(z), \tag{A15}$$

where $I_0(z)$ is the modified Bessel function, which is expanded as

$$I_0(z) = 1 + \frac{z^2}{4} + \frac{z^4}{64} + O(z^6). \tag{A16}$$

Concerning the ρ integration, we first employ the decomposition,

$$\frac{\rho^4}{(2\xi^2+\rho^2)^2} = 1 - \frac{4\xi^2}{2\xi^2+\rho^2} + \frac{4\xi^4}{(2\xi^2+\rho^2)^2}, \tag{A17}$$

and then apply the formulas like

$$\int_0^\infty d\rho\rho e^{-\frac{2\rho^2}{\Delta}} I_0\left(\frac{4\rho l_i}{\Delta}\right) = \frac{\Delta}{4} e^{\frac{2l_i^2}{\Delta}}. \tag{A18}$$

The final result is

$$B = \left(\frac{N}{N_0}\right)^4 \frac{g_1}{\pi\Delta}\left[\frac{1}{4} + d_7 e^{-\frac{2l_i^2}{\Delta}}\right], \tag{A19}$$

where d_7 is given in Table I.

APPENDIX B: CALCULATION OF INTERVORTICE INTERACTION U

This interaction is due to the integral,

$$\begin{aligned}
U &\equiv g_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_i^2 f_j^2 dxdy \\
&= \frac{g_2}{\pi^2\Delta^2}\left(\frac{N}{N_0}\right)^4 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy e^{-\frac{2(x^2+y^2)}{\Delta}} \\
&\times \frac{(x-x_i)^2+(y-y_i)^2}{2\xi^2+(x-x_i)^2+(y-y_i)^2} \\
&\times \frac{(x-x_j)^2+(y-y_j)^2}{2\xi^2+(x-x_j)^2+(y-y_j)^2}.
\end{aligned} \tag{B1}$$

With use of a prescription $U \equiv \frac{g_2}{\pi^2\Delta^2}(\frac{N}{N_0})^4\hat{U}$, the integral becomes

$$\begin{aligned}
\hat{U} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy e^{-\frac{2(x^2+y^2)}{\Delta}} \\
&\times \left[1 - 2\xi^2\left(\frac{1}{2\xi^2+(x-x_i)^2+(y-y_i)^2} + (i \to j)\right)\right. \\
&+ \left.\frac{4\xi^4}{(2\xi^2+(x-x_i)^2+(y-y_i)^2)(i\to j)}\right] \\
&\equiv \hat{U}_0 + \hat{U}_2 + \hat{U}_4.
\end{aligned} \tag{B2}$$

$\hat{U}_0$, $\hat{U}_2$, and $\hat{U}_4$ corresponds to the contributions from $O(\xi^0)$, $O(\xi^2)$, and $O(\xi^4)$, respectively. Among them, $\hat{U}_4$ is responsible to the vortex-vortex interaction, which we shall calculate below.

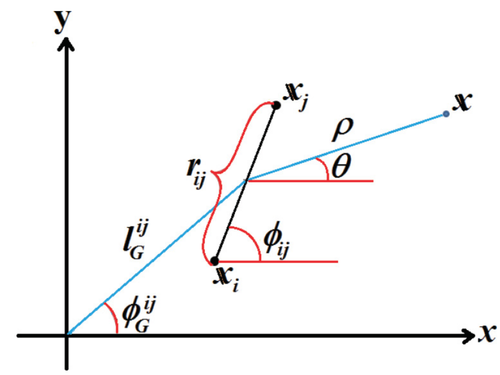

FIG. 3. (Color online) New integration variables, center-of-mass coordinates, and relative coordinates.

Let us define the center-of-mass and relative coordinates by

$$x_G^{ij} = \frac{x_j + x_i}{2}; \quad y_G^{ij} = \frac{y_j + y_i}{2}, \tag{B3}$$

and

$$x_i - x_j = r_{ij}\cos\phi_{ij}; \quad y_i - y_j = r_{ij}\sin\phi_{ij}, \tag{B4}$$

respectively, and transform the integration variables to new ones as (see Fig. 3)

$$\begin{aligned}
x - x_G^{ij} &= \rho\cos\theta, \quad x_G^{ij} = l_G^{ij}\cos\phi_G^{ij}, \\
y - y_G^{ij} &= \rho\sin\theta, \quad y_G^{ij} = l_G^{ij}\sin\phi_G^{ij}, \\
dxdy &= \rho d\rho d\theta.
\end{aligned} \tag{B5}$$

Then $\hat{U}_4$ in Eq. (B2) becomes

$$\begin{aligned}
\hat{U}_4 = &\int_0^\infty \rho d\rho \int_0^{2\pi} d\theta' e^{\frac{-2}{\Delta}((l_G)^2+\rho^2+2\rho l_G^{ij}\cos(\theta'+\phi_{ij}-\phi_G^{ij}))} \\
&\times \left[\frac{4\xi^4}{\left(2\xi^2+\rho^2+\frac{r_{ij}^2}{4}\right)^2 - \rho^2 r_{ij}^2\cos^2(\theta')}\right],
\end{aligned} \tag{B6}$$

where we moved to a new angle variable $\theta' \equiv \theta - \phi_{ij}$. Using the expansion,

$$e^{-X\cos(\theta'+\alpha)} = \sum_{n=-\infty}^{\infty}(-1)^n I_n(X) e^{in\alpha} e^{in\theta'}, \tag{B7}$$

in Eq. (B6) and keeping the $n = 0$ term, the integration over the angle variable leads to

$$\int_0^{2\pi} d\theta' \frac{4\xi^4}{\left(2\xi^2+\rho^2+\frac{r_{ij}^2}{4}\right)^2 - \rho^2 r_{ij}^2\frac{1+\cos 2\theta'}{2}} = \frac{8\pi\xi^4}{2\xi^2+\rho^2+\frac{r_{ij}^2}{4}} \frac{1}{\sqrt{\left(2\xi^2+\left(\rho-\frac{r_{ij}}{2}\right)^2\right)\left(2\xi^2+\left(\rho+\frac{r_{ij}}{2}\right)^2\right)}}, \tag{B8}$$

where we used the formula $\int_0^{2\pi} d\theta \frac{1}{c+b\cos\theta} = \frac{2\pi}{\sqrt{(c-b)(c+b)}}$.

We shall proceed to ρ integration. Here we should note the following: In the case $\xi \ll 1$, the Lorentzian-like function on the right-hand side of Eq. (B8) is sharply peaked around $\rho = \frac{r_{ij}}{2}$ and is well approximated for $\rho > 0$ by Gaussian, $\frac{8\sqrt{2}\pi\xi^3}{r_{ij}^3}\exp(-\frac{(\rho-\frac{r_{ij}}{2})^2}{4\xi^2})$. Therefore, the ρ integration leads to

$$\begin{aligned}
\hat{U}_4 &= \frac{8\sqrt{2}\pi\xi^3}{r_{ij}^3} e^{-\frac{2(l_G)^2}{\Delta}}\left[\int_0^\infty \rho e^{-\frac{2\rho^2}{\Delta}} I_0\left(\frac{4\rho l_G^{ij}}{\Delta}\right)\exp\left(-\frac{\left(\rho-\frac{r_{ij}}{2}\right)^2}{4\xi^2}\right)d\rho\right] \\
&= \frac{8\pi^{3/2}\xi^4}{(1+\xi^2/\Delta)(1+8\xi^2/\Delta)^{1/2}} \frac{1}{r_{ij}^2} I_0\left(\frac{2l_G^{ij} r_{ij}}{\Delta(1+2\xi^2/\Delta)}\right)\exp\left(-\frac{2l_G^2}{\Delta}\right)\exp\left(-\frac{r_{ij}^2}{2\Delta(1+8\xi^2/\Delta)}\right),
\end{aligned} \tag{B9}$$

where the integration was carried out by the saddle-point approximation which is justified in the case $\xi \ll 1$. Since each vortex dynamics occurs in the range $\frac{2l_G^{ij} r_{ij}}{\Delta} \ll 1$ because of the notion above Eq. (16) in Sec. II, we may approximate $I_0(\frac{2l_G^{ij} r_{ij}}{\Delta(1+2\xi^2/\Delta)}) \sim 1$ and neglect the contributions from $I_n(x)$ with $n = 1,2,\ldots$ in the region $x \ll 1$. Then we reach

$$U_4 \equiv U_4\left(r_{ij}, l_G^{ij}\right) \equiv \frac{g_2}{\pi^2\Delta^2}\left(\frac{N}{N_0}\right)^4 \hat{U}_4 \approx \frac{8g_2\xi^4}{\pi^{1/2}\Delta^2}\frac{1}{r_{ij}^2}\exp\left(-\frac{r_{ij}^2 + 4\left(l_G^{ij}\right)^2}{2\Delta}\right), \tag{B10}$$

with use of $(\frac{N}{N_0})^4 \approx 1$. At first the magnitude of the intervortices interaction looks very small [i.e., $O(\xi^4)$], but, after scaling to make unity the inertial mass of each vortex, it becomes $O(g_2)$ [see Eq. (17)]. It is interesting that the intervortice interaction energy in a multicomponent BEC with no trap estimated in a different approximation (i.e., under the Abrikosov ansatz) is also proportional to $\frac{1}{r_{ij}^2}$ in the asymptotic region [31].

Finally we note the remaining contributions, $\hat{U}_0 + \hat{U}_2$, in Eq. (B2). Their integration is quite simple and gives rise to

$$U_0 + U_2 \equiv \frac{g_2}{\pi^2\Delta^2}\left(\frac{N}{N_0}\right)^4 (\hat{U}_0 + \hat{U}_2) \approx \frac{g_2}{\pi\Delta}\left(\frac{N}{N_0}\right)^4\left[\frac{1}{2} + 2\frac{\xi^2}{\Delta}\left(\gamma + \ln\frac{2\xi^2}{\Delta}\right)\left(e^{-\frac{l_i^2}{\Delta}} + e^{-\frac{l_j^2}{\Delta}}\right)\right]. \tag{B11}$$

In the final expression, the first term is constant, giving no contribution to the dynamics in Eq. (13), and the second one is a sum of single-particle contributions of $O(\frac{\xi^2}{\Delta})$ which renormalizes the 7th line in Eq. (8), giving no substantial contribution to Eq. (13).

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